Graph Complexes and their Cohomology

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Abstract

In this thesis we study cohomologies of two kind of graph complexes, Kontsevich graph complexes and hairy graph complexes, by three different methods. In the first method we combinatorially calculate the dimensions of Kontsevich graph complexes by providing generating functions, leading to the Euler characteristics. Secondly, we prove that multiple edges may be omitted without essentially changing the cohomology in both type of complexes. The third method introduces extra differentials on all complexes, leading to spectral sequences that converge to (almost) zero with the standard differential being the first one. This leads to the conclusion that classes of standard cohomology come in pairs.

Zusammenfassung

In dieser Dissertation untersuchen wir Kohomologien von zwei Arten der Graphenkomplexe, Kontsevich Graphenkomplexe und Graphenkomplexe von Graphen mit “Haaren”, mit drei verschiedenen Methoden. Mit der ersten Methode berechnen wir kombinatorisch die Dimensionen von Kontsevich Graphenkomplexe durch Bereitstellen der erzeugenden Funktionen, was zur Euler-Charakteristik führt. Zweitens zeigen wir, dass Graphen mit mehreren Kanten in beiden Komplexen verzichtet werden kann, ohne die Kohomologie wesentlich zu verändern. Die dritte Methode führt zusätzliche Differentiale auf allen Komplexen ein. Dies führt zu Spektralsequenzen, deren erste Seite standard Differential hat, und die gegen (fast) null konvergieren. Dies führt zu dem Schluss, dass Klassen der Graphenkohomologie in Paaren auftreten.

Contents

Chapter 1

Introduction

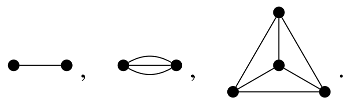
1.1 Motivation

The graph complexes in its various kinds are some of the most intriguing objects of homological algebra. Each of them plays a certain role in a subfield of homological algebra or algebraic topology. They have an elementary and simple combinatorial definition, yet we know almost nothing about what their cohomology actually is.

Generally speaking, graph complexes are graded vector spaces of formal linear combinations of isomorphism classes of some kind of graphs, with the standard differential defined by vertex splitting (or, dually, edge contraction). There are many different kinds of graph complexes that are connected to many fields in mathematics. In this thesis we study two kinds of them: Kontsevich graph complexes and hairy graph complexes.

The most basic graph complexes are introduced by Maxim Kontsevich in his study of the formality conjecture [11] in 1997, and in some form also earlier in [9]. These complexes come in versions GC_n, where n ranges over integers (see Chapter 2 for the definition). Physically, GC_n is formed by vacuum Feynman diagrams of a topological field theory in dimension n. Alternatively, GC_n governs the deformation theory of E_n operads in algebraic topology. More precisely, it governs the deformation theory of algebraic operads e_n that are quasi-isomorphic to E_n. See Section 2.14 or [21] for more details about connection between graph complexes and operads.

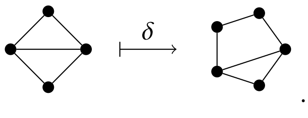
Some examples of graphs in Kontsevich graph complexes are:



The differential on this complex is taking the Lie bracket with Maurer-Cartan element

$\lambda :=$.

Roughly speaking, it splits the vertex into two vertices, connects them with a new edge, and reconnects edges that were connected to the previous vertex to the two new vertices in all possible ways, summing over all vertices. For example:



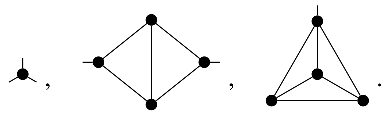
There are also negative signs that come from some splitting, leading to some graphs canceling each other. A precise definition is described in Chapter 2.

The other kind of graph complexes we consider are *hairy graph complexes*. They are spanned by graphs with external legs ("hairs"). These complexes come in versions $\mathrm{HGC}_{m,n}$ where m and n range over integers (see Chapter 2 for the definition). They compute the rational homotopy of the spaces of embeddings modulo immersions $\overline{\mathrm{Emb}}(\mathbb{R}^m, \mathbb{R}^n)$, provided that $n-m \geq 3$, see [6], cf. [1]. Here, $\mathrm{Emb}(\mathbb{R}^m, \mathbb{R}^n)$ is the space of "generalized long knots", i.e. smooth maps $\mathbb{R}^m \to \mathbb{R}^n$ that are embeddings (image is diffeomorphic to $\mathbb{R}^m$) and fixed outside a compact subset of $\mathbb{R}^m$, and $\overline{\mathrm{Emb}}(\mathbb{R}^m, \mathbb{R}^n)$ is the space of generalized long knots modulo immersions, i.e the homotopy fiber of the inclusion

$$\mathrm{Emb}(\mathbb{R}^m, \mathbb{R}^n) \to \mathrm{Imm}(\mathbb{R}^m, \mathbb{R}^n).$$

Immersions Imm are differentiable functions whose derivative is everywhere injective. Unlike embeddings, immersions does not need to be injective. Smale-Hirsch theorem implies that the space of immersions $\mathrm{Imm}(\mathbb{R}^m, \mathbb{R}^n)$ is homotopic to the space of Grassmannians $\mathrm{Grass}_{m,n}$, that are well understood. Therefore this result gives us more understanding of all embeddings $\mathbb{R}^m \to \mathbb{R}^n$. For more details see [13, 16, 2, 1]. The result is still in preparation in [6]. Connection to operads described in Section 2.14 is again the first step of the proof.

Some examples of hairy graphs are:



The differential is again splitting a vertex, similar to the one in Kontsevich graph complexes. A precise definition is described in Chapter 2.

Both kinds of complexes split into the product of subcomplexes with fixed loop order and the hairy graph complex splits into the product of subcomplexes with fixed number of hairs, cf. (2.10). Furthermore, the complexes GC_n and $\mathrm{GC}_{n'}$, respectively $\mathrm{HGC}_{m,n}$ and $\mathrm{HGC}_{m',n'}$, are isomorphic up to some degree shifts of those subcomplexes if $m \equiv m' \bmod 2$ and $n \equiv n' \bmod 2$, cf. (2.11). Hence it suffices to understand two possible cases of GC_n and 4 possible cases of $\mathrm{HGC}_{m,n}$ according to the parity of m and n.

Although graph complexes are simple objects easy to define, their cohomology is still largely unknown. The long standing open problem we are attacking in this thesis is the following:

Open Problem: Compute the cohomologies $H(\mathrm{GC}_n)$ and $H(\mathrm{HGC}_{m,n})$.

Very little is known about those cohomologies, and a very few tools are available to compute them. The most non-trivial result is that $H^0(\mathrm{GC}_2)$ is isomorphic to Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_1$, shown in [21]. There are also some results connecting hairy and non-hairy complexes, see [19, 17]. In this thesis we are using some techniques to improve our knowledge of these cohomologies, but we are still far away from the full understanding.

1.2 Overview of the thesis

Kontsevich graph complexes and hairy graph complexes are historically introduced separately. Here, in order not to repeat similar definitions, we define the full hairy complexes $\mathrm{fHGC}_{m,n}$, and say that the full Kontsevich graph complexes fGC_n are sub-complexes spanned by hairy graphs with zero hairs. The definition is elementary and purely combinatorial. Also, many technical subcomplexes and subspaces of them will be needed in the thesis. They are all defined in Chapter 2 "Definitions and known results". In its sections 2.12 and 2.13 some previous results we will use in the thesis are listed. The last Section 2.14 gives us another approach to the graph complexes, using operads. This definition is more elegant, yet less elementary. It also gives us a connection between graph complexes and operads, motivating our interest in graph complexes.

Chapter 3 is the most independent from the rest of the thesis. Here we calculate the dimensions of (various sub-complexes of) the full Kontsevich graph complexes for a fixed number of edges and vertices. Since the differential does not play a role here, we actually give formulas for dimensions of broader class of graph spaces, see Definition 2.3, cf. Definition 2.5. For graph complexes, this gives us Euler characteristics of them, being an information of its cohomology. Formulas are still complicated, but useful for computer calculations. In the last Section 3.8 we give dimensions and Euler characteristics calculated by the computer. These results are from author's first paper in the field of graph complexes [22] written with his adviser Thomas Willwacher. Here, the result is extended to classes of graph spaces that do not form complexes.

In Chapter 4 we prove that the cohomology (almost) does not change if we disallow multiple hairs and multiple edges. Together with earlier results that the cohomology (almost) does not change if we disallow tadpoles and vertices are bound to be at least 3-valent (Propositions 2.22 and 2.24, [21, Proposition 3.4]), this result implies that it is essentially irrelevant which sub-complex we consider, and it is enough to consider smallest versions of graph complexes. This result is also from the author's first paper with Thomas Willwacher [22], but the result is here extended to hairy graph complexes.

In the remaining chapters 5–10 we deal with extra differentials. The basic idea is to deform the standard differential δ to $\delta + \delta^{extra}$ making the complex (almost) acyclic. The extra piece does not fix the loop order or the number of hairs, and a spectral sequence can be found (from the loop order filtration or the number of hairs filtration) such that the standard differential is the first differential. Therefore, on the first page of the spectral sequence we see the standard cohomology we are interested in, and because the whole differential is acyclic, classes of it cancel with each other on further pages, cf. Table 5.1. Therefore, classes come in pairs.

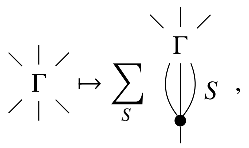
Recall that we are dealing with 6 complexes, Kontsevich graph complex for two parities of n and hairy graph complex for 4 combinations of parities of m and n. The exact extra differentials may be very different in those different cases. The essential difficulty is to prove that deformed differential is (almost) acyclic, and that has to be done for every extra differential separately.

Chapter 5 introduces an extra differential on GC_n for n even. The extra differential here is adding an edge in all possible ways. This result is from the paper [7] of Anton Khoroshkin, Thomas Willwacher and the author, slightly improved by a result from [23] of the author alone. In Section 6 we introduce deformed differential on GC_n for n odd. Instead of taking Lie bracket with λ, it is presented by taking Lie bracket with extended Maurer-Cartan element m' (6.2). The result is solely from [7]. There, in Appendix A, one can find a conceptually more satisfying explanation of the same statement, using operads.

On every hairy graph complex we have two different extra differentials. This gives us even more insight into its cohomology because two extra differentials group classes in pairs in two different ways. This leads us to the "waterfall mechanism" explained in Section 7.2.

For each hairy complex one extra differential is not the subject of this thesis. The results are cited in Section 2.13. We have two of them: for m and n of the same parity it is adding a hair in all possible ways, and for m and n of the opposite parity it is taking a Lie bracket with an element ω (2.37).

The other extra differentials on hairy graph complexes are subject of this thesis. There are again two of them. Chapter 7 deals with the case when m is even. The extra differential here is "grouping hairs":



where the sum runs over all subsets S of the set of hairs with at least two elements, see Definition 7.1. The result is from the paper [8] again of Anton Khoroshkin, Thomas Willwacher and the author.

The extra differential for m odd is described in that paper, but its acyclicity is only conjectured. It is transforming a hair into an edge towards all other vertices. A proof that it makes the deformed differential (almost) acyclic is found in the author's paper [23]. The proof is a bit lengthy and slightly different for two parities of n. Therefore we split it here into different chapters: Chapter 9 deals with n odd, Chapter 10 deals with n even and in Chapter 8 before them we introduce some notions and results needed in both cases.

Chapter 2

Definitions and known results

In this chapter we recall the definitions of M. Kontsevich's graph complexes and hairy graph complexes. We go through both definitions simultaneously because they are similar. Auxiliary graph complexes DGC_n and WGC_n needed in Chapter 6 are defined there, in order not to make the definition too messy.

It is convenient to define graphs and graph spaces in a bit broader sense than needed for the complexes, because the formulas of Chapter 3 can be easily adopted to all of them, and they may be useful.

2.1 Basic Notation

We work over a ground field $\mathbb{K}$ of characteristic zero. All vector spaces and differential graded vector spaces are assumed to be $\mathbb{K}$-vector spaces. The phrase differential graded will be abbreviated by dg. We use cohomological conventions, so that the degree of the differentials is $+1$.

We denote the subspace of elements of homogeneous degree k of a graded vector space V by $\mathrm{D}_k V$. We define the degree shifted vector space $V[r]$ such that $\mathrm{D}_k V[r] \cong \mathrm{D}_{k+r} V$.

2.2 Graph spaces

There are many standard definitions of a graph. We use the following.

Definition 2.1. Let $v > 0$, $e \geq 0$ and $h \geq 0$ be integers. Let $V := \{1, 2, \ldots, v\}$ be set of vertices, $E := \{1, 2, \ldots, e\}$ set of edges and $H := \{1, 2, \ldots, h\}$ set of hairs.

A *graph* Γ with v vertices, e edges and h hairs is an ordered pair (Γ_E, Γ_H) of maps $\Gamma_E : E \to V^2$ and $\Gamma_H : H \to V$. We say that edge $a \in E$ connects vertices $(\Gamma_E)_1(a)$ and $(\Gamma_E)_2(a)$ in that direction, and that a hair $k \in H$ is on the vertex $\Gamma_H(k)$.

For $a \in E$ we say that vertices $(\Gamma_E)_1(a)$ and $(\Gamma_E)_2(a)$ are connected. We extend the notion of being connected by transitivity, such that it is a relation of equivalence. Equivalence classes are called *connected components*. A graph is *connected* if it has one connected component and *disconnected* if it has more than one connected component.

Note that by our definition a graph has distinguishable vertices, edges and hairs, and edges are directed.

Definition 2.2. For $v > 0$, $e \geq 0$, $h \geq 0$ let $\mathrm{gra}_{v,e,h}$ be the set of all graphs with v vertices, e edges and h hairs.

For $v > 0$, $e \geq 0$, $h \geq 0$ the *graph space*

$$\bar{\mathrm{V}}_v \bar{\mathrm{E}}_e \bar{\mathrm{H}}_h \mathrm{GS}$$

is the vector space of formal direct sums of elements from $\mathrm{gra}_{v,e,h}$ over the field $\mathbb{K}$.

Letters in front of the main label GS indicate the number of elements of the graph (vertices, edges or hairs), and the bars above them indicate that elements are numbered, i.e. distinguishable. They can be written in any order, i.e. $\bar{\mathrm{V}}_v \bar{\mathrm{E}}_e \bar{\mathrm{H}}_h \mathrm{GS} = \bar{\mathrm{E}}_e \bar{\mathrm{H}}_h \bar{\mathrm{V}}_v \mathrm{GS}$.

Let S_n be a symmetric group of order n. There are natural actions of S_v, S_e and S_h on $\mathrm{gra}_{v,e,h}$, that permute the vertices V, edges E and hairs H respectively. The group $S_2^{\otimes e}$ acts on $\mathrm{gra}_{v,e,h}$ by switching the directions of edges.

There is a total action of the semi-direct product $S_e \ltimes S_2^{\otimes e}$ on edges. All actions are extended to a representation on $\bar{\mathrm{V}}_v\bar{\mathrm{E}}_e\bar{\mathrm{H}}_h\mathrm{GS}$.

Let sgn_n^+ be the trivial one-dimensional representation and let sgn_n^- be the (one-dimensional) *sign representation* of S_n. In that representation even permutations send $v \mapsto v$ and odd permutations send $v \mapsto -v$. For $n = 0$ or 1 both representations are the same.

Definition 2.3. For any space of the form $\bar{\mathrm{V}}_v\mathrm{S}$ (e.g. $\bar{\mathrm{V}}_v\bar{\mathrm{E}}_e\bar{\mathrm{H}}_h\mathrm{GS}$) and for $\rho \in \{+,-\}$ we define a space

$$\mathrm{V}_v^\rho\mathrm{S} := \left(\bar{\mathrm{V}}_v\mathrm{S} \otimes \mathrm{sgn}_v^\rho\right)^{S_v},$$

where the group in the superscript means taking the space of invariants of the action.

Similarly, for any space of the form $\bar{\mathrm{E}}_e\mathrm{S}$ (e.g. $\bar{\mathrm{E}}_e\bar{\mathrm{V}}_v\bar{\mathrm{H}}_h\mathrm{GS} = \bar{\mathrm{V}}_v\bar{\mathrm{E}}_e\bar{\mathrm{H}}_h\mathrm{GS}$) and for $\mu, \nu \in \{+,-\}$ we define a space

$$\mathrm{E}_e^{\mu\nu}\mathrm{S} := \left(\bar{\mathrm{E}}_e\mathrm{S} \otimes \mathrm{sgn}_e^\mu \otimes \left(\mathrm{sgn}_2^\nu\right)^{\otimes e}\right)^{S_e \ltimes S_2^{\times e}}.$$

Finally, for any space of the form $\bar{\mathrm{H}}_h\mathrm{S}$ (e.g. $\bar{\mathrm{H}}_h\bar{\mathrm{V}}_v\bar{\mathrm{E}}_e\mathrm{GS}$) and for $\kappa \in \{+,-\}$ we define a space

$$\mathrm{H}_h^\kappa\mathrm{S} := \left(\bar{\mathrm{H}}_h\mathrm{S} \otimes \mathrm{sgn}_h^\kappa\right)^{S_h}.$$

Since the groups are finite and we are working over a field of characteristic zero, the space of invariants is isomorphic to the space of co-invariants. Therefore, equivalent definition with co-invariants may be used.

The definition can be recursively used to define spaces

$$\mathrm{V}_v^\rho\mathrm{E}_e^{\mu\nu}\mathrm{H}_h^\kappa\mathrm{GS}.$$

It is easily shown that the order of taking invariants does not matter.

With a slightly abuse of notation, the isomorphism class of a graph will also be called a graph. For distinguishing, any linear combination (or series) of graphs will not be called a graph, but an element of a graded graph space, or graph complex.

Example 2.4. For $\rho = \mu = \nu = \kappa = +$ tensoring with sgn^+ does not make any difference so taking the space of co-invariants simply makes elements indistinguishable and

$$\mathrm{V}_v^+\mathrm{E}_e^{++}\mathrm{H}_h^+\mathrm{GS}$$

is the space spanned by graphs with v indistinguishable vertices, e indistinguishable undirected edges and h indistinguishable hairs.

2.3 Graded graph spaces

To construct a graph complex from graph spaces, first we need to make a graded space. By slightly abusing notation, by the word complex we will sometimes mean graded graph space, without determining the differential.

The precise grading depends on two integers, m and n. The choice of signs also depend on those numbers, precisely on their parity, as shown in the following definition.

Definition 2.5. Let $m, n \in \mathbb{Z}$. We define

$$\mathrm{V}_v\mathrm{E}_e\mathrm{H}_h\mathrm{fHGC}_{m,n} := \begin{cases} \mathrm{V}_v^+\mathrm{E}_e^{-+}\mathrm{H}_h^-\mathrm{GS} & \text{for } m \text{ even, } n \text{ even,} \\ \mathrm{V}_v^-\mathrm{E}_e^{+-}\mathrm{H}_h^+\mathrm{GS} & \text{for } m \text{ even, } n \text{ odd,} \\ \mathrm{V}_v^+\mathrm{E}_e^{-+}\mathrm{H}_h^+\mathrm{GS} & \text{for } m \text{ odd, } n \text{ even,} \\ \mathrm{V}_v^-\mathrm{E}_e^{+-}\mathrm{H}_h^-\mathrm{GS} & \text{for } m \text{ odd, } n \text{ odd} \end{cases} \quad [m - vn - (1-n)e - (m+1-n)h].$$

Note that, for fixed m and n the degree of the graph depends on the number of vertices, edges and hairs. As we see, the degree of a graph with v vertices, e edges and h hairs is $d = -m + vn + (1-n)e + (m+1-n)h$. We say that the degree of a vertex is n, the degree of an edge is $1-n$ and the degree of a hair is $m+1-n$.

Definition 2.6. For graded spaces of the form $\mathrm{V}_v\mathrm{S}$ for $v \geq 1$ (e.g. $\mathrm{V}_v\mathrm{E}_e\mathrm{H}_h\mathrm{fHGC}$) we define a space

$$\mathrm{S} := \prod_{v \geq 1} \mathrm{V}_v\mathrm{S}.$$

Similarly, for graded spaces of the form $\mathrm{E}_e\mathrm{S}$ for $e \geq 0$ (e.g. $\mathrm{E}_e\mathrm{H}_h\mathrm{fHGC}$) we define a space

$$\mathrm{S} := \prod_{e\geq 0} \mathrm{E}_e\mathrm{S}.$$

Finally, for graded spaces of the form $\mathrm{H}_h\mathrm{S}$ for $h \geq 0$ we define a space

$$\mathrm{S} := \prod_{h\geq 0} \mathrm{H}_h\mathrm{S}.$$

The spaces

$$\mathrm{fHGC}_{m,n}$$

are called *full hairy graph complexes*.

2.4 Prefixes to the graph spaces

Normally we work with the graded spaces $\mathrm{fHGC}_{m,n}$ or its interesting sub-spaces defined later on. Prefixes V, E and H are used if we want to look the sub-space spanned by graphs for the fixed number of vertices, edges or hairs. In the following definition we introduce more prefixes that may be useful.

Definition 2.7. The graded space

$$\mathrm{C}_c\mathrm{S} \tag{2.1}$$

is the sub-space of S spanned by classes of graphs with c connected components.

Let

$$\mathrm{B}_b\mathrm{S} := \prod_{b=e-v} \mathrm{V}_v\mathrm{E}_e\mathrm{S}, \tag{2.2}$$

$$\mathrm{A}_a\mathrm{S} := \prod_{a=e+h} \mathrm{E}_e\mathrm{H}_h\mathrm{S}, \tag{2.3}$$

$$\mathrm{F}_f\mathrm{S} := \prod_{f=e+h-v} \mathrm{V}_v\mathrm{E}_e\mathrm{H}_h\mathrm{S}, \tag{2.4}$$

$$\mathrm{P}_p\mathrm{S} := \prod_{p=2c-e+v} \mathrm{C}_c\mathrm{V}_v\mathrm{E}_e\mathrm{S}. \tag{2.5}$$

We also use indices in the prefix of the form of inequality $\mathrm{X}_{\geq k}$, that obviously means the sub-complex spanned by graphs which fulfil the inequality, e.g.

$$\mathrm{V}_{\geq v}\mathrm{S} = \prod_{k\geq v} \mathrm{V}_k\mathrm{S}. \tag{2.6}$$

For any prefix X we define

$$\mathrm{X}_{even}\mathrm{S} := \prod_{k \text{ even}} \mathrm{X}_k\mathrm{S}, \quad \mathrm{X}_{odd}\mathrm{S} := \prod_{k \text{ odd}} \mathrm{X}_k\mathrm{S}, \tag{2.7}$$

$$\mathrm{X}_{<f,par}\mathrm{S} := \prod_{\substack{k<f \\ k \text{ and } f \text{ of the same parity}}} \mathrm{X}_k\mathrm{S}. \tag{2.8}$$

Note that for any prefix X (that is not the degree prefix D) a graded space always splits as the direct product

$$\mathrm{S} = \prod_{k\in\mathbb{Z}} \mathrm{X}_k\mathrm{S}.$$

Therefore every vector $\Gamma \in \mathrm{S}$ can be uniquely split as

$$\Gamma = \sum_{k\in\mathbb{Z}} \mathrm{X}_k\Gamma,$$

where $\mathrm{X}_k\Gamma \in \mathrm{X}_k\mathrm{S}$. We will use prefixes also to denote this part of a vector.

We summarize all prefixes in the following table. Let v be the number of vertices in a graph, e the number of edges, h the number of hairs, c the number of connected components, and $d = -m + vn + (1-n)e + (m+1-n)h$ its degree.

Prefix	Explanation
D_d	Graphs of degree d
V_k	Graphs with $v = k$
E_k	Graphs with $e = k$
H_k	Graphs with $h = k$
C_k	Graphs with $c = k$
B_k	Graphs with $e - v = k$
A_k	Graphs with $e + h = k$
F_k	Graphs with $e + h - v = k$
P_k	Graphs with $2c - e + v = k$

Table 2.1: Prefixes to the graph complexes used in this thesis.

2.5 The standard differential

On graph spaces we define $\delta : \bar{\mathrm{V}}_v\bar{\mathrm{E}}_e\bar{\mathrm{H}}_h\mathrm{GS} \mapsto \bar{\mathrm{V}}_{v+1}\bar{\mathrm{E}}_{e+1}\bar{\mathrm{H}}_h\mathrm{GS}$:

$$\delta(\Gamma) = \sum_{x=1}^{v}\left(\frac{1}{2}s_x(\Gamma) - a_x(\Gamma)\right) - \sum_{k=1}^{h} e_k(\Gamma), \tag{2.9}$$

where:

- s_x stands for "splitting of x" and means inserting $\overset{x}{\bullet}\xrightarrow{e+1}\overset{v+1}{\bullet}$ instead of the vertex x and summing over all possible ways of connecting edges and hairs that have been connected to x to vertices x and $v+1$;

- a_x stands for "adding an edge at x" and means adding $\overset{x}{\bullet}\xrightarrow{e+1}\overset{v+1}{\bullet}$ on the vertex x, while all edges and hairs stay on vertex x;

- e_k stands for "extracting hair k" and means adding $\overset{x}{\bullet}\xrightarrow{e+1}\overset{v+1}{\bullet}\nwarrow_k$ on the vertex x where the hair was instead of the hair k, while all other hairs and edges stay on vertex x.

The differential induces the differential on graded spaces $\mathrm{fHGC}_{m,n}$.

Remark 2.8. The following claims are easy to check.

- The map δ induces the map $\delta : \mathrm{V}^{\rho}_{v}\mathrm{E}^{\mu\nu}_{e}\mathrm{H}^{\kappa}_{h}\mathrm{GS} \to \mathrm{V}^{\rho}_{v+1}\mathrm{E}^{\mu\nu}_{e+1}\mathrm{H}^{\kappa}_{h}\mathrm{GS}$ for all choices of ρ, μ, ν and κ.

- The four cases of signs in Definition 2.5 are the only cases when δ is not trivially zero and $\delta^2 = 0$. Our degree convention makes δ of degree 1, so it is a differential and $(\mathrm{fHGC}_{m,n}, \delta)$ is a complex.

- Unless x is an isolated vertex with no or one hair, a_x and h_k for k being the hair on x will cancel two or four simple terms of the splitting s_x.

The differential does not change the number $b = e - v$ and the number of hairs h, so the complexes split as

$$(\mathrm{fHGC}_{m,n}, \delta) = \prod_{b\in\mathbb{Z}} (\mathrm{B}_b\mathrm{fHGC}_{m,n}, \delta) = \prod_{h\geq 0} (\mathrm{H}_h\mathrm{fHGC}_{m,n}, \delta) = \prod_{b\in\mathbb{Z}} \prod_{h\geq 0} (\mathrm{B}_b\mathrm{H}_h\mathrm{fHGC}_{m,n}, \delta)\,, \tag{2.10}$$

Definition 2.5 implies that for $m - m'$ and $n - n'$ even signs of representations are the same and it holds that

$$(\mathrm{B}_b\mathrm{H}_h\mathrm{fHGC}_{m,n}, \delta) = (\mathrm{B}_b\mathrm{H}_h\mathrm{fHGC}_{m',n'}, \delta)\,[(n - n')(b + 1) - (m - m' - n + n')h]. \tag{2.11}$$

Therefore, finding the cohomology of $(\mathrm{fHGC}_{m,n}, \delta)$ for one choice of m and n immediately gives the cohomology of $(\mathrm{fHGC}_{m',n'}, \delta)$ if m' and n' are of the same parity as m and n respectively. So, it is enough to investigate four cases of m and n to understand all of them.

2.6 Kontsevich graph complex

Definition 2.9. For $n \in \mathbb{Z}$ the *Kontsevich full graph complex*, or simply *full graph complex* is

$$\mathrm{fGC}_n := \mathrm{H}_0\mathrm{fHGC}_{n,n}.$$

It is simply the hairless version of the complex. Note that in that case the number m is equal to n and it matters only for the degree shift. It holds

$$\mathrm{H}_0\mathrm{fHGC}_{m,n} = \mathrm{fGC}_n[m - n]. \tag{2.12}$$

The standard differential is simpler: for a graph $\Gamma \in \mathrm{fGC}_n$ with v vertices

$$\delta(\Gamma) = \sum_{x=1}^{v} \left(\frac{1}{2} s_x(\Gamma) - a_x(\Gamma)\right). \tag{2.13}$$

2.7 Lie algebra structure on Kontsevich graph complex

Definition 2.10. A *pre-Lie algebra* $(V, \triangleleft)$ is a vector space V with a bilinear map $\triangleleft : V \otimes V \to V$, called *pre-Lie product*, satisfying the relation

$$(x \triangleleft y) \triangleleft z - x \triangleleft (y \triangleleft z) = (x \triangleleft z) \triangleleft y - x \triangleleft (z \triangleleft y).$$

A *graded Lie algebra* $(V, [\cdot,\cdot])$ is a graded vector space V with a bilinear map $[\cdot,\cdot] : V \otimes V \to V$, called *Lie product*, satisfying the following:

- $[\cdot,\cdot]$ respects the gradation of V:
$$[\mathrm{D}_i V, \mathrm{D}_j V] \subset \mathrm{D}_{i+j} V,$$
- for $x \in \mathrm{D}_i V$ and $y \in \mathrm{D}_j V$
$$[x, y] = -(-1)^{ij}[y, x],$$
- for $x \in \mathrm{D}_i V$, $y \in \mathrm{D}_j V$, and $z \in \mathrm{D}_k V$
$$(-1)^{ik}[x, [y, z]] + (-1)^{ij}[y, [z, x]] + (-1)^{jk}[z, [x, y]] = 0 \quad \text{(graded Jacobi itentity)}.$$

Lemma 2.11. *Let $(V, \triangleleft)$ be a pre-Lie algebra that respects the graduation, i.e. $\mathrm{D}_i V \triangleleft \mathrm{D}_j V \subset \mathrm{D}_{i+j} V$. Then V with the map*

$$[x, y] = x \triangleleft y - (-1)^{ij} y \triangleleft x,$$

where $x \in \mathrm{D}_i V$ and $y \in \mathrm{D}_j V$, is a Lie-product.

Proof. Easy verification. □

Definition 2.12. Let $\Gamma_1, \Gamma_2 \in \mathrm{fGC}_n$ be two graphs. The pre-Lie product on fGC_n is defined on graphs as

$$\Gamma_1 \triangleleft \Gamma_2 = \sum_{x \in V(\Gamma_1)} \pm \Gamma_1 \triangleleft_x \Gamma_2 = \sum_{x \in V(\Gamma_1)} \Gamma_1 \underset{x}{\equiv} \Gamma_2 \,.$$

Here $V(\Gamma_1)$ is the set of vertices in Γ_1. In the each term a vertex $x \in V(\Gamma_1)$ is replaced by the whole graph Γ_2 and $\Gamma_1 \underset{x}{\equiv} \Gamma_2$ denotes the sum of all possible ways edges from Γ_1 connected to x can be reconnected to vertices of Γ_2. To define the sign precisely, before the action of symmetric groups, we first rename x to be the last vertex, delete it, and name in the new graph all vertices and edges of Γ_2 after those of Γ_1, preserving the order.

The Lie bracket is

$$[\Gamma_1, \Gamma_2] = \Gamma_1 \triangleleft \Gamma_2 - (-1)^{|\Gamma_1||\Gamma_2|} \Gamma_2 \triangleleft \Gamma_1,$$

where $|\Gamma|$ is the degree of Γ.

Lemma 2.13. *The map* $\triangleleft$ *on* fGC_n *is well defined and it is indeed a pre-Lie product and therefore* $[\cdot, \cdot]$ *is a Lie product.*

Proof. Easy verification. □

Note that the Lie product is of the degree 0.

Remark 2.14. There is a Maurer-Cartan element

$$\lambda = \bullet\!\!-\!\!\bullet, \tag{2.14}$$

i.e., $[\lambda, \lambda] = 0$ and $|\lambda| = 1$. One checks that the standard differential on Kontsevich graph complex is indeed

$$\delta = \frac{1}{2}[\lambda, \cdot]. \tag{2.15}$$

2.8 General notation of graph complexes

It is time to discuss our general notation for graph complexes. Except for some technical complexes, it is of the form

$$\mathrm{pNe}_{\mathrm{i}}^{\mathrm{sup}}, \tag{2.16}$$

meaning the following. For completeness we provide the full list of possibilities, and the reference to the definition later in the thesis.

- The first small letter p indicates the general type of the complex. It is used for denoting various important subcomplexes of the full complex, and is omitted in the most important complexes. If stated, it means:
  - f - full complex,
  - b - bounded complex (see Definition 8.12),
  - u - unbounded complex (see Definition 8.12).
- Capital letters N indicate the main type of complex. It can be:
  - GC - graph complex,
  - DGC - dotted graph complex, defined and used in Section 6.2,
  - WGC - waved graph complex, defined and used in Section 6.4,
  - HGC - hairy graph complex.
- The ending e deals with connectivity and if stated means (see Definition 2.20):
  - c - connected graphs,
  - d - disconnected graphs,
  - n - graphs without separate vertex.

- The subscript i defines degrees and parity (1 number for ordinary complexes and 2 numbers for hairy complexes).
- The superscript sup adds additional constraints on graphs that are allowed to form the complex, see Section 2.9 below. It is a combination of no or some constrains from the following:
  - $\geq i$ - graph that have all vertices at least i-valent for $i \in \{1, 2, 3\}$, or † or ‡, see Definition 2.16,
  - $\not\circlearrowleft$ - graphs without tadpoles,
  - $\not\Leftrightarrow$ - graphs without multiple edges,
  - $\not\succ$ - graphs without multiple hairs.

  The other use of the superscript is adding an extra graph to the complex: if GS is a graded graph space and $\Gamma \notin \mathrm{GS}$ is a graph, usually simple one, then $\mathrm{GS}^\Gamma := \mathrm{GS} \oplus \mathbb{K}\Gamma$. See e.g. (6.5).

2.9 Additional constraints on graphs

Definition 2.15. The valence of a vertex $x \in V$ is the number of edges $a \in E$ such that $(\Gamma_E)_1(a) = x$ plus the number of edges $a \in E$ such that $(\Gamma_E)_2(a) = x$ plus the number of hairs $k \in H$ such that $\Gamma_H(k) = x$.

An edge a such that $\Gamma_E(a) = (x, x)$ for some vertex $x \in V$ is called a *tadpole* on vertex x.

A pair of vertices $\{x, y\} \subset V$ is a *multiple edge* if there are more than one edges $a \in E$ such that $\Gamma_E(a) = (x, y)$ or (y, x).

A vertex x is said to have *multiple hair* if there are more than one hairs $k \in H$ such that $\Gamma(k) = x$.

Definition 2.16. For every graph complex GS we denote by

- $\mathrm{GS}^{\geq i}$ the space or complex spanned by (classes of) graphs that have all vertices at least i-valent for $i \in \{1, 2, 3\}$,
- $\mathrm{GS}^{\dagger}$ the subspace of $\mathrm{GS}^{\geq 1}$ spanned by (classes of) graphs that do not have λ as a connected component,
- $\mathrm{GS}^{\ddagger}$ the subspace of $\mathrm{GS}^{\geq 2}$ spanned by (classes of) graphs that do not have 2-valent vertex with a hair,
- $\mathrm{GS}^{\not\circlearrowleft}$ the space or complex spanned by (classes of) graphs without tadpoles,
- $\mathrm{GS}^{\not\Leftrightarrow}$ the space or complex spanned by (classes of) graphs without multiple edges,
- $\mathrm{GS}^{\not\succ}$ the space or complex spanned by (classes of) graphs without multiple hairs.

Remark 2.17. For every graph complex GS the inclusions hold:

$$\mathrm{GS}^{\geq 3} \subset \mathrm{GS}^{\ddagger} \subset \mathrm{GS}^{\geq 2} \subset \mathrm{GS}^{\dagger} \subset \mathrm{GS}^{\geq 1} \subset \mathrm{GS}.$$

The reason why we introduced two special constraints is the elegant Corollary 2.26 used later in the thesis.

Remark 2.18. Let us consider the space

$$\mathrm{S} := \mathrm{V}_v^\rho \mathrm{E}_e^{-\nu} \mathrm{H}_h^\kappa \mathrm{GS} = \left(\bar{\mathrm{E}}_e \mathrm{V}_v^\rho \mathrm{H}_h^\kappa \mathrm{GS} \otimes \mathrm{sgn}_e^- \otimes \left(\mathrm{sgn}_2^\nu\right)^{\otimes e} \right)^{S_e \ltimes S_2^{\times e}} .$$

If there is a multiple edge in the graph $\Gamma \in \mathrm{gra}_{v,e,h}$ that represents the element of S, switching two edges of the multiple edge turns the graph Γ into itself. Switching two edges is an action of an odd permutation in S_e, so taking co-invariants identify $\Gamma \equiv \Gamma$, implying $\Gamma \equiv 0$. So, graph with multiple edges do not contribute at all and

$$\mathrm{V}_v^\rho \mathrm{E}_e^{-\nu} \mathrm{H}_h^\kappa \mathrm{GS} = \mathrm{V}_v^\rho \mathrm{E}_e^{-\nu} \mathrm{H}_h^\kappa \mathrm{GS}^{\not\Leftrightarrow}.$$

Similarly one shows that

$$\mathrm{V}_v^\rho \mathrm{E}_e^{\mu -} \mathrm{H}_h^\kappa \mathrm{GS} = \mathrm{V}_v^\rho \mathrm{E}_e^{\mu -} \mathrm{H}_h^\kappa \mathrm{GS}^{\not\circlearrowleft},$$

$$\mathrm{V}_v^\rho \mathrm{E}_e^{\mu\nu} \mathrm{H}_h^- \mathrm{GS} = \mathrm{V}_v^\rho \mathrm{E}_e^{\mu -} \mathrm{H}_h^- \mathrm{GS}^{\not\succ}.$$

Remark 2.19. For every graph that satisfies the constraint defined above the graph obtained by the differential satisfies the same constraint. Therefore, all complexes with constraints are well defined.

2.10 Graph complexes by connectivity

Definition 2.20. For every complex denoted by $\mathrm{pN}_{\mathrm{i}}^{\mathrm{sup}}$ let

$$\mathrm{pNc}_{\mathrm{i}}^{\mathrm{sup}} := \mathrm{C}_1\mathrm{pN}_{\mathrm{i}}^{\mathrm{sup}}, \quad (\textit{connected complex}),$$

$$\mathrm{pNd}_{\mathrm{i}}^{\mathrm{sup}} := \mathrm{C}_{\geq 2}\mathrm{pN}_{\mathrm{i}}^{\mathrm{sup}}, \quad (\textit{disconnected complex}).$$

A separating vertex in a connected graph is a vertex x that, if removed with all its adjacent edges and hairs, splits a graph into a disconnected graph. Complex

$$\mathrm{pNn}_{\mathrm{i}}^{\mathrm{sup}}$$

is the sub-complex of $\mathrm{pNc}_{\mathrm{i}}^{\mathrm{sup}}$ spanned by graphs without separating vertex, called *non-separated complex*.

Every graph can be seen as the union of its connected components, and the standard differential affects only the components. Therefore, after being careful with degrees, for every $m, n \in \mathbb{Z}$ and superscript sup it holds that

$$\left(\mathrm{fHGC}_{m,n}^{s}, \delta\right) = S^+\left(\mathrm{fHGCc}_{m,n}^{s}[-m], \delta\right)[m], \tag{2.17}$$

where S^+ is the symmetric product:

$$S^+(V) := \prod_{c\geq 1}\left(V^{\otimes c}\right)^{S_c}, \tag{2.18}$$

where the symmetric group in the superscript means taking the space of co-invariants.

The same is true for the Kontsevich complex:

$$(\mathrm{fGC}_n^{s}, \delta) = S^+\left(\mathrm{fGCc}_n^{s}[-n], \delta\right)[n]. \tag{2.19}$$

The cohomology of the complex commutes with the symmetric product:

$$H\left(S^+(\mathrm{GS})\right) = S^+(H(\mathrm{GS})). \tag{2.20}$$

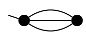
Therefore, it is enough to compute the cohomology of connected complexes to understand the whole cohomology.

2.11 Special graphs

For the reader's convenience in this section we collect the notation for all special graphs used in this thesis.

Definition 2.21. For two graphs Γ and Γ', the graph $\Gamma \cup \Gamma'$ is the disconnected graph that consists of Γ and Γ'. For the matter of sign, all vertices, edges and hairs of the first graph come before those of the second one. For a graph Γ and $n \geq 1$, $\Gamma^{\cup n}$ is the graph that consist of n copies of the graphs Γ put together.

(2.21) $\Theta :=$  $\in \mathrm{fGC}_n$,

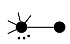
(2.22) $\Theta_h :=$  $\in \mathrm{fHGC}_{m,n}$,

(2.23) $\Phi :=$ $\in \mathrm{fHGC}_{m,n}$.

A *star* is

(2.24) $\sigma_a :=$  $\in \mathrm{fHGC}_{m,n}$

with $a \geq 0$ hairs. Let

(2.25) $\lambda_a :=$  $\in \mathrm{fHGC}_{m,n}$

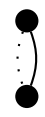
with $a-1$ hairs on one vertex for $a \geq 1$. Note that for $m-n$ even there can be at most one hair on a vertex, so in that case $\sigma_a = 0$ for $a \geq 2$ and $\lambda_a = 0$ for $a \geq 3$. Moreover, let

(2.26) $\sigma := \sigma_0 =$ $\in \mathrm{fGC}_n, \quad \lambda := \lambda_1 =$ $\in \mathrm{fGC}_n$,

$$(2.27)\qquad m := \sum_{j\geq 2}\frac{1}{j!}$$  ^{j} $$= \sum_{j\geq 1}\frac{1}{(2j+1)!}$$  $^{2j+1} \in \mathrm{fGC}_n, \quad m' := m+\lambda = \sum_{j\geq 0}\frac{1}{(2j+1)!}$  $^{2j+1} \in \mathrm{fGC}_n,$

Where the bold line with a number j next to it means j simple lines. Furthermore,

$$(2.28)\qquad c := \sum_{j>0}\frac{j}{(2j+1)!}$$  $^{2j+1} \in \mathrm{fGC}_n,$

(2.29) $p :=$  $\in \mathrm{DGC}_1, \quad \zeta :=$  $\in \mathrm{DGC}_1,$

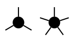
(2.30) $\alpha := \sum_{n\geq 1}\frac{1}{n!}\sigma_1^{\cup n} =$  $+ \frac{1}{2}$  $+ \frac{1}{3!}$  $+ \cdots \in \mathrm{fHGC}_{m,n},$

$$(2.31)\qquad \omega := \sum_{k\geq 1}\frac{1}{(2k+1)!}\sigma_{2k+1} \in \mathrm{fHGC}_{m,n}.$$

For $j \geq 1$

$$(2.32)\qquad \Sigma_j := \sum_{\substack{k_i\geq 0\\ \sum_i ik_i = j}}\prod_i \frac{(-1)^{k_i}}{k_i!((2i+1)!)^{k_i}}\bigcup_i \sigma_{2i+1}^{\cup k_i} \in \mathrm{fHGC}_{m,n}.$$

For example,

$\Sigma_3 = \frac{-1}{6\cdot 6^3}$  $+ \frac{1}{6\cdot 5!}$  $+ \frac{-1}{7!}$ .

2.12 Known results

In this section we list the simple results that relate cohomologies of similar complexes.

Proposition 2.22 ([21, Proposition 3.4], partially contained in [9, 10])**.**

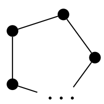
$$H(\mathrm{fGCc}_n,\delta) = H\left(\mathrm{fGCc}_n^{\geq 3},\delta\right) \oplus \bigoplus_{\substack{j\geq 1\\ j\equiv 2n+1 \mod 4}} \mathbb{K}[n-j],$$

$$H\left(\mathrm{fGCc}_n^{\text{⚲}},\delta\right) = H\left(\mathrm{fGCc}_n^{\geq 3},\delta\right) \oplus \bigoplus_{\substack{j\geq 3\\ j\equiv 2n+1 \mod 4}} \mathbb{K}[n-j],$$

$$H\left(\mathrm{fGCc}_n^{\geq 3},\delta\right) = H\left(\mathrm{fGCc}_n^{\geq 3\,\text{⚲}},\delta\right).$$

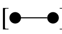
The same relations hold with a constraint .

Listed classes in degree $j-n$ are loop classes with j 2-valent vertices and j edges:

(2.33) .

The following results are straightforward.

Proposition 2.23.

$H\left(\mathrm{fGCc}_n^{\geq 1},\delta\right) = H(\mathrm{fGCc}_n,\delta) \oplus \mathbb{K}[$  $],$

$$H\left(\mathrm{fGCc}_n^{\geq 2},\delta\right) = H\left(\mathrm{fGCc}_n^{\dagger},\delta\right) = H(\mathrm{fGCc}_n,\delta).$$

The same relations hold with constraints ⚲ *or* .

The propositions imply that it is enough to understand the cohomology of $\mathrm{fGCc}_n^{\geq 3}$. Because of its importance, we introduce the new, shorter notation for it:

$$\mathrm{GC}_n := \mathrm{fGCc}_n^{\geq 3}, \tag{2.34}$$

and we call it the *graph complex*.

In [21] Proposition 2.22 has been proven for hairless graphs. But almost the same proof works if we allow hairs, so exactly the same proposition with HGC instead of GC holds. Note that in $\mathrm{fHGC}_{m,n}$ loops are of degree $j-m$. We list the easy results.

Proposition 2.24.

$$H\left(\mathrm{fHGCc}_{m,n}, \delta\right) = H\left(\mathrm{fHGCc}_{m,n}^{\geq 3}, \delta\right) \oplus \bigoplus_{\substack{j\geq 1 \\ j\equiv 2n+1 \mod 4}} \mathbb{K}[m-j],$$

$$H\left(\mathrm{fHGCc}_{m,n}^{\not\circlearrowleft}, \delta\right) = H\left(\mathrm{fHGCc}_{m,n}^{\geq 3}, \delta\right) \oplus \bigoplus_{\substack{j\geq 3 \\ j\equiv 2n+1 \mod 4}} \mathbb{K}[m-j],$$

$$H\left(\mathrm{fHGCc}_{m,n}^{\geq 3}, \delta\right) = H\left(\mathrm{fHGCc}_{m,n}^{\geq 3 \not\circlearrowleft}, \delta\right).$$

The same relations hold with constraints $\not\Leftrightarrow$ *or* $\not\multimap$.

Proposition 2.25.

$$H\left(\mathrm{fHGCc}_{m,n}^{\geq 1}, \delta\right) = H\left(\mathrm{fHGCc}_{m,n}, \delta\right) \oplus \mathbb{K}[\lambda],$$

$$H\left(\mathrm{fHGCc}_{m,n}^{\ddagger}, \delta\right) = H\left(\mathrm{fHGCc}_{m,n}^{\geq 2}, \delta\right) = H\left(\mathrm{fHGCc}_{m,n}^{\dagger}, \delta\right) = H\left(\mathrm{fHGCc}_{m,n}, \delta\right).$$

The same relations hold with constraints $\not\circlearrowleft$, $\not\Leftrightarrow$ *or* $\not\multimap$.

Corollary 2.26.

$$H\left(\mathrm{fHGC}_{m,n}^{\ddagger}, \delta\right) = H\left(\mathrm{fHGC}_{m,n}^{\geq 2}, \delta\right) = H\left(\mathrm{fHGC}_{m,n}^{\dagger}, \delta\right) = H\left(\mathrm{fHGC}_{m,n}, \delta\right).$$

The same relations hold with constraints $\not\circlearrowleft$, $\not\Leftrightarrow$ *or* $\not\multimap$.

Proof. Since the differential does not change the number of connected components, this is an easy consequence of the previous proposition and Relations (2.17) and (2.20). □

We may be interested in the strictly hairy complex $\mathrm{H}_{\geq 1}\mathrm{HGC}_{m,n}^{\mathrm{sup}}$. It holds that

$$\left(\mathrm{fHGC}_{m,n}^{\mathrm{sup}}, \delta\right) = \left(\mathrm{fGC}_n^{\mathrm{sup}}, \delta\right)[m-n] \oplus \left(\mathrm{H}_{\geq 1}\mathrm{fHGC}_{m,n}^{\mathrm{sup}}, \delta\right), \tag{2.35}$$

implying the following corollary.

Corollary 2.27.

$$H\left(\mathrm{H}_{\geq 1}\mathrm{fHGCc}_{m,n}^{\geq 3 \not\circlearrowleft}, \delta\right) = H\left(\mathrm{H}_{\geq 1}\mathrm{fHGCc}_{m,n}^{\geq 3}, \delta\right) = H\left(\mathrm{H}_{\geq 1}\mathrm{fHGCc}_{m,n}^{\ddagger \not\circlearrowleft}, \delta\right) = H\left(\mathrm{H}_{\geq 1}\mathrm{fHGCc}_{m,n}^{\ddagger}, \delta\right) =$$
$$= H\left(\mathrm{H}_{\geq 1}\mathrm{fHGCc}_{m,n}^{\geq 2 \not\circlearrowleft}, \delta\right) = H\left(\mathrm{H}_{\geq 1}\mathrm{fHGCc}_{m,n}^{\geq 2}, \delta\right) = H\left(\mathrm{H}_{\geq 1}\mathrm{fHGCc}_{m,n}^{\dagger \not\circlearrowleft}, \delta\right) = H\left(\mathrm{H}_{\geq 1}\mathrm{fHGCc}_{m,n}^{\dagger}, \delta\right) =$$
$$= H\left(\mathrm{H}_{\geq 1}\mathrm{fHGCc}_{m,n}^{\geq 1 \not\circlearrowleft}, \delta\right) = H\left(\mathrm{H}_{\geq 1}\mathrm{fHGCc}_{m,n}^{\geq 1}, \delta\right) = H\left(\mathrm{H}_{\geq 1}\mathrm{fHGCc}_{m,n}^{\not\circlearrowleft}, \delta\right) = H\left(\mathrm{H}_{\geq 1}\mathrm{fHGCc}_{m,n}, \delta\right).$$

The same relations hold with constraints $\not\Leftrightarrow$ *or* $\not\multimap$.

Therefore it is enough to understand the cohomology of one of the complexes, namely $\mathrm{H}_{\geq 1}\mathrm{fHGCc}_{m,n}^{\geq 3}$. We introduce the new, shorter notation for it:

$$\mathrm{HGC}_{m,n} := \mathrm{H}_{\geq 1}\mathrm{fHGCc}_{m,n}^{\geq 3}, \tag{2.36}$$

and we call it the *hairy graph complex*.

As we have seen, the results equally hold if we add constraint $\not\Leftrightarrow$ or $\not\multimap$ in all superscripts. The question is what is the relation between cohomologies of complexes with and without those constraints. It is solved in the second part of this thesis, Chapter 4.

The following result relates the non-separated complexes to the other complexes.

Proposition 2.28 ([4], cf. Appendix F of [21][1])**.** *For every constraint* $\mathrm{sup} \subset \{\geq i, \not\circlearrowleft, \not\Leftrightarrow, \not\multimap\}$, $i \in \{1,2,3\}$

$$H\left(\mathrm{fHGCn}_n^{\mathrm{sup}}, \delta\right) = H\left(\mathrm{fHGCc}_n^{\mathrm{sup}}, \delta\right).$$

[1]Note that in these references the result is only shown for one version of the graph complex. However, the proofs do not depend on the presence or absence of hairs, tadpoles or multiple edges, and on minimal valence of vertices.

2.13 An extra differential

A lot about extra differentials for various graph complexes and their use will be explained later in Sections 5–10. Here we mention one extra differential that was not part of author's research, see [8, Theorem 4], c.f. [18, 17, 21].

First we consider the case when $m-n$ is even. As a representative of that case we work with a complex $\mathrm{fHGC}_{n,n}$.

Definition 2.29. Let $\chi^1 : \mathrm{fHGC}_{n,n} \to \mathrm{fHGC}_{n,n}$ be the map that adds a hair in all possible ways. Let on a graph Γ with v vertices

$$\chi(\Gamma) = (-1)^v \chi^1(\Gamma).$$

Proposition 2.30. *On* $\mathrm{fHGC}_{n,n}$ *it holds that*

1. $\chi^2 = 0$,
2. $\chi\delta + \delta\chi = 0$,
3. $(\delta + \chi)^2 = 0$.

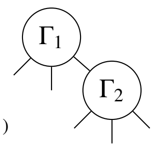
On $\mathrm{fHGC}_{n,n}$ χ is of degree 1, so the proposition implies that χ and $\delta + \chi$ are differentials. They are also differentials on the connected part $\mathrm{fHGCc}_{n,n}$.

Theorem 2.31 ([8, Theorem 4], c.f. [18, 17, 21][2])**.** *The complex* $(\mathrm{fHGCc}_{n,n}, \delta + \chi)$ *is acyclic. The spectral sequence on the number of hairs converges correctly. All cancellations except those of loop classes* (2.33) *are on the second page.*

Secondly we consider the case when $m-n$ is odd with $\mathrm{fHGC}_{n-1,n}$ being the representative. To do that we need to introduce Lie-algebra structure on $\mathrm{fGC}_{m,n}$. It is not an extension of Lie algebra on non-hairy graphs, but rather a different operation.

Definition 2.32. Let $\Gamma_1, \Gamma_2 \in \mathrm{fHGC}_{m,n}$ be two graphs. Pre-Lie product on $\mathrm{fHGC}_{m,n}$ is obtained by summing over all ways of attaching one hair of Γ to a vertex of Γ':

$$\Gamma_1 \bullet \Gamma_2 = \sum_{x \in H(\Gamma_1)}$$



The Lie bracket is

$$[\Gamma_1, \Gamma_2] = \Gamma_1 \bullet \Gamma_2 - (-1)^{|\Gamma_1||\Gamma_2|} \Gamma_2 \bullet \Gamma_1,$$

where $|\Gamma|$ is the degree of Γ.

Let σ_a be the graph consisting of one vertex and a hairs and let

$$\omega := \sum_{k \geq 1} \frac{1}{(2k+1)!} \sigma_{2k+1}. \tag{2.37}$$

Proposition 2.33. *In* $(\mathrm{fHGC}_{n-1,n}, \delta)$ ω *is Maurer-Cartan element, i.e.* $\delta\omega + \frac{1}{2}[\omega, \omega] = 0$.

The proposition implies that $\delta + [\omega, \cdot]$ is another differential on $\mathrm{fHGC}_{m,n}$. It is also a differential on connected part $\mathrm{fHGCc}_{m,n}$.

Theorem 2.34 ([8][Theorem 4], c.f. [18, 17, 21])**.** *There is a quasi-isomorphism*

$$\mathbb{K} \oplus \left(\mathrm{fGCc}_n^{\geq 2}, \delta\right) \to \left(\mathrm{HGC}_{n-1,n}, \delta + [\omega, \cdot]\right).$$

The spectral sequence on the number of hairs converges correctly. Classes of $H(\mathrm{GC}_n)$ *(non-loop classes) represented by* Γ *are sent to classes represented by* $\chi(\Gamma) \in \mathrm{H}_1\mathrm{HGC}_{n-1,n}$.

[2] In those papers non-hairy and hairy complexes are strictly separated, while here fHGC contains also a non-hairy part. Therefore, the result in those papers is expressed as an existence of a quasi-isomorphism between non-hairy and hairy graph complex. Note also that the definition in those papers has one more graph making one class of cohomology.

2.14 Operadic definition of graph complexes

The definition of graph complexes we gave here is the most elementary. There is a slightly more elegant definition through graph operads. In this section we give a sketch of this definition and some results. It gives us also a connection between graph complexes and graph operads that may be further connected to other type of operads (e.g. topological operads) or other mathematical objects, as a motivation to study them. This section is not essential in understanding what follows in the thesis, and therefore may be skipped. We here assume that the reader is slightly familiar with operads and cooperads. A standard textbook for that is [14] by J.-L. Loday and B. Vallette. We are following notation from [21].

For an operad $\mathcal{P}$, we denote its space of arity $N \in \mathbb{N}_0$ by $\mathcal{P}(N)$. The *operadic r-fold desuspension* is an operad $\mathcal{P}\{r\}$ such that

$$\mathcal{P}\{r\}(N) = \mathcal{P}(N) \otimes \operatorname{sgn}_N^{\otimes r}[(N-1)r] \tag{2.38}$$

where sgn_N is the sign representation of S_N. Identical notation is used for cooperads.

Among standard operads we will use the commutative operad Com, Lie operad Lie and the operads e_n governing n-algebras for $n = 2, 3, \ldots$, see [14, Chapter 13]. Com is generated by an associative operation $\cdot \wedge \cdot$ with trivial action and Lie is generated by $[\cdot,\cdot]$ with trivial action that fulfills Jacobi identity. We will need graded operad Lie_n similar to Lie, but with the generator $[\cdot,\cdot]$ of degree $n-1$ and with sign sgn_2^- action if n is odd. Operad e_n is generated by both $\cdot \wedge \cdot$ of degree 0 and $[\cdot,\cdot]$ of degree $n-1$ that fulfill and act as in the operads Com and Lie, with additional operation $[a \wedge b, c] = a \wedge [b,c] + [a,c] \wedge b$. There are natural embeddings $\mathsf{Com} \to e_n$ and $\mathsf{Lie}_n \to e_n$. All those operads are considered differential graded (dg) operads with zero differential.

For the matter of defining deformation complex we will need their generalizations hoCom, hoLie and hoe_n. For an operad $\mathcal{P}$, $\mathsf{ho}\mathcal{P}$ is quasi-free operad (free as an operad, but differential does not act freely) quasi-isomorphic to $\mathcal{P}$, and it is minimal resolution of $\mathcal{P}$. It is technically constructed as

$$\mathsf{ho}\mathcal{P} = \Omega(\mathcal{P}^\vee), \tag{2.39}$$

where $\mathcal{P}^\vee$ is Koszul dual of $\mathcal{P}$ (see [14, 7.2]) and for a coaugmented cooperad C, $\Omega(C)$ is obtained by cobar construction (see [14, 7.3.2]). It holds (see [14, Chapter 13]) that

$$\mathsf{Com}^\vee = \mathsf{Lie}^*\{1\}, \quad \mathsf{Lie}^\vee = \mathsf{Com}^*\{1\}, \quad e_n^\vee = e_n^*\{n\}, \tag{2.40}$$

where $\mathcal{P}^*$ is the (standard) dual cooperad of an operad $\mathcal{P}$.

For a cooperad C finite dimensional in each degree and an operad $\mathcal{P}$ let

$$\operatorname{Hom}_S(C, \mathcal{P}) := \prod_{N \geq 0} \operatorname{Hom}_{S_N}(C(N), \mathcal{P}(N)) = \prod_{N \geq 0} C(N)^* \otimes_{S_N} \mathcal{P}(N), \tag{2.41}$$

where S_N in the superscript means taking the space of co-invariance under the action of the symmetric group on the inputs. Therefore, $C(N)^* \otimes_{S_N} \mathcal{P}(N)$ has indistinguishable inputs, but those of C and $\mathcal{P}$ are glued together. There is a natural Lie-algebra structure on $C(N)^* \otimes_{S_N} \mathcal{P}(N)$ that comes from operadic composition and the algebra is called *convolution Lie algebra*. If the cooperad C and the operad $\mathcal{P}$ are dg, the convolution Lie algebra is also dg. For more details see [14, Section 6.4.2].

Let us suppose that we have a map of dg operads $f : \Omega(C) \to \mathcal{P}$. Such a map bijectively determines a Maurer-Cartan element α in the dg Lie algebra $\operatorname{Hom}_S(C, \mathcal{P})$. We may twist the differential by this Maurer-Cartan element to obtain a dg Lie algebra which we denote

$$\operatorname{Def}\left(\Omega(C) \xrightarrow{f} \mathcal{P}\right) \tag{2.42}$$

and call the *deformation complex* of the map $f : \Omega(C) \to \mathcal{P}$. The corresponding notation in [14, Section 6.4.9] is $\operatorname{Hom}_S^\alpha(C, \mathcal{P})$.

We now define a graph operad gra_n for arity N:

$$\mathsf{Gra}_n(N) := \prod_{e \geq 0} \begin{cases} \bar{\mathrm{V}}_N \mathrm{E}_e^{-+} \mathrm{H}_0 \mathrm{GS} & \text{for } n \text{ even,} \\ \bar{\mathrm{V}}_N \mathrm{E}_e^{+-} \mathrm{H}_0 \mathrm{GS} & \text{for } n \text{ odd} \end{cases} [-(1-n)e]. \tag{2.43}$$

In the other words, in arity N there are graphs with N vertices that represent inputs, and edges are indistinguishable up to some parity. In the operadic composition $\Gamma \circ_j \Gamma'$ the whole graph Γ' is inserted instead of the vertex j of the graph Γ and edges that were connected to j are connected to vertices of Γ' in all possible ways. The differential is zero.

Here, degree shift and signs are chosen so that there is an injective map

$$e_n \to \mathsf{Gra}_n, \tag{2.44}$$

defined on generators as follows.

$$\cdot \wedge \cdot \mapsto \underset{1}{\circ} \quad \underset{2}{\circ}\,, \qquad\qquad [\cdot,\cdot] \mapsto \underset{1}{\circ}\!\!-\!\!\underset{2}{\circ}\,. \tag{2.45}$$

By restriction, one obtains maps $\mathsf{Lie}_n \to \mathsf{Gra}_n$, and hence also maps $\mathsf{hoLie}_n \to \mathsf{Gra}_n$. The full graph complex fGC_n is defined as deformation complex of the latter map:

$$\mathrm{fGC}_n := \mathrm{Def}(\mathsf{hoLie}_n \to \mathsf{Gra}_n). \tag{2.46}$$

Since $\mathsf{hoLie}_n = \Omega(\mathsf{Lie}_n^\vee) = \Omega(\mathsf{Com}^*\{n\})$, as a vector space fGC_n is

$$\prod_{N\geq 0} \mathsf{Com}\{-n\}(N) \otimes_{S_N} \mathsf{Gra}(N) = \prod_{N\geq 0} \mathsf{Gra}(N)\{-n\}_{S_N} = \prod_{N\geq 0} \left(\mathsf{Gra}(N) \otimes \mathrm{sgn}_N^{\otimes n}[-(N-1)n]\right)_{S_N},$$

because Com is 1-dimensional in each arity. One checks that this is exactly the space defined in Definition 2.9 through Definitions 2.5 and 2.6. Lie algebra structure defined in Definition 2.12 comes naturally from the operadic composition. Maurer-Cartan element of the map $\mathsf{hoLie}_n \to \mathsf{Gra}_n$ is exactly $\lambda = \bullet\!\!-\!\!\bullet$, so the differential is exactly (2.13), c.f. (2.15).

There is an operation called *operadic twisting*, which, from an operad $\mathcal{P}$ with a map $\mathsf{Lie}_n \to \mathcal{P}$, produces another operad $Tw\mathcal{P}$, with a map $\mathsf{Lie}_n \to Tw\mathcal{P}$. The general theory of operadic twisting is described in [5]. Here we apply the twisting functor to the operad Gra_n with the standard map $\mathsf{Lie}_n \to \mathsf{Gra}_n$ and obtain an operad

$$\mathsf{fGraphs}_n := Tw\mathsf{Gra}_n. \tag{2.47}$$

This operad is introduced by Kontsevich in his proof of the formality of the little cubes operads [12]. Generators of $\mathsf{fGraphs}_n(N)$ (as a vector space) can be depicted by graphs with two kinds of vertices:

- "external" vertices, which are numbered $1,\dots,N$ and
- "internal" vertices, which are indistinguishable.

In pictures we draw external vertices white and internal vertices black, see Figure 2.1. The operadic composition is obtained by insertion at external vertices, similarly to the composition in Gra_n. Here there is a non-zero differential, the one that splits internal vertices like in graph complexes. Note that there is a natural inclusion $\mathsf{Gra}_n \to \mathsf{fGraphs}$, so there are natural maps $e_n \to \mathsf{fGraphs}$ and $\mathsf{Lie}_n \to \mathsf{fGraphs}$.

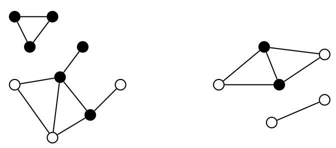


Figure 2.1: A typical graph from $\mathsf{fGraphs}_n(3)$ (left) and one from $\mathsf{Graphs}_n(4) \subset \mathsf{fGraphs}_n(4)$ (right).

Let $\mathsf{Graphs} \subset \mathsf{fGraphs}$ be the sub-operad spanned by graphs with all internal vertices at least 3-valent, and with no connected component consisting entirely of internal vertices. There is again an inclusion $\mathsf{Gra}_n \to \mathsf{Graphs}$ and maps $e_n \to \mathsf{Graphs}$ ans $\mathsf{Lie}_n \to \mathsf{Graphs}$.

Here, we can consider the complex

$$\mathrm{Def}(\mathsf{hoLie}_m \to \mathsf{Graphs}_n). \tag{2.48}$$

It is similar to GC_n but has two kind of vertices, internal of degree n and external of degree m. The differential on internal vertices comes from the differential on the operad Graphs_n and the differential on the external vertices comes from the map $\mathsf{hoLie}_m \to \mathsf{Graphs}_n$.

The hairy graph complex $\mathrm{HGC}_{m,n}$ can be understood as a sub-complex of $\mathrm{Def}(\mathsf{hoLie}_m \to \mathsf{Graphs}_n)$ spanned by connected graphs with all external vertices 1-valent. External vertices together with an edge towards them are understood as hairs.

Chapter 3

Calculating the dimensions of Graph vector spaces

In this chapter we calculate the dimension of hairless graph spaces

$$\dim^{\rho\mu\nu\mathrm{sup}}_{v,e} := \dim\left(\mathrm{V}^{\rho}_{v}\mathrm{E}^{\mu\nu}_{e}\mathrm{H}_0\mathrm{GS}^{\mathrm{sup}}\right) \tag{3.1}$$

for all signs ρ, μ, ν and all superscripts $\mathrm{sup} \in \{{}^{\geq i}, {}^{\not\circlearrowleft\geq i}, {}^{\not\multimap\geq i}, {}^{\not\circlearrowleft\not\multimap\geq i}\}$, $i \in \{0, 1, 2, 3\}$. This is the extension of the result from [22, Section 3] where we calculated those dimensions only for two possibilities of signs that appear in complexes (c.f. Definition 2.5). It leads to calculating the Euler characteristics of complexes $\mathrm{B}_b\mathrm{fGC}^{\mathrm{sup}}_n$.

Here we will get the result for all possibilities of signs, since there is no essential difference. In particular, the case $\rho = \mu = \nu = +$ is independent from graph spaces and can be done already with sets $\mathrm{gra}^{\mathrm{sup}}_{v,e,0}$. It simply calculates the number of graphs, so the result may be helpful in general.

The case with hairs would not be essentially different either. We leave it for a future research if found useful.

3.1 General calculation

It holds that

$$\dim^{\rho\mu\nu\mathrm{sup}}_{v,e} = \dim\left(\left(\bar{\mathrm{V}}_{v}\mathrm{E}^{\mu\nu}_{e}\mathrm{H}_0\mathrm{GS}^{\mathrm{sup}} \otimes \mathrm{sgn}^{\rho}_{v}\right)^{S_v}\right) = \frac{1}{|S_v|}\sum_{g\in S_v} \xi^{\rho\mu\nu\mathrm{sup}}_{v,e}(g), \tag{3.2}$$

where

$$\xi^{\rho\mu\nu\mathrm{sup}}_{v,e} \tag{3.3}$$

is the character of the representation of S_v on $\bar{\mathrm{V}}_{v}\mathrm{E}^{\mu\nu}_{e}\mathrm{H}_0\mathrm{GS}^{\mathrm{sup}} \otimes \mathrm{sgn}^{\rho}_{v}$. Furthermore, it is enough to calculate characters of conjugacy classes of S_v, i.e. on partitions $\underline{j} = (j_1, j_2, \dots)$, $\sum_\alpha \alpha j_\alpha = v$:

$$\dim^{\rho\mu\nu\mathrm{sup}}_{v,e} = \frac{1}{v!}\sum_{\substack{j_1,j_2,\dots\geq 0\\ \sum_\alpha \alpha j_\alpha = v}} \frac{v!}{\prod_\alpha j_\alpha! \alpha^{j_\alpha}} \xi^{\rho\mu\nu\mathrm{sup}}_{v,e}(g_{\underline{j}}), \tag{3.4}$$

where $g_{\underline{j}}$ is any element of the conjugacy class $\underline{j}$ of S_v. Let

$$P^{\rho\mu\nu\mathrm{sup}} = \sum_{v,e\geq 0} \dim^{\rho\mu\nu\mathrm{sup}}_{v,e}\, s^v t^e \tag{3.5}$$

be the generating functions. Then

$$P^{\rho\mu\nu\mathrm{sup}} = \sum_{v,e\geq 0}\sum_{\substack{j_1,j_2,\dots\geq 0\\ \sum_\alpha \alpha j_\alpha = v}} \frac{1}{\prod_\alpha j_\alpha! \alpha^{j_\alpha}} \xi^{\rho\mu\nu\mathrm{sup}}_{v,e}(g_{\underline{j}}) s^v t^e = \sum_{j_1,j_2,\dots\geq 0}\left(\prod_{\alpha\geq 1}\frac{s^{\alpha j_\alpha}}{j_\alpha! \alpha^{j_\alpha}}\right)\xi^{\rho\mu\nu\mathrm{sup}}_{\underline{j}} \tag{3.6}$$

where

$$\xi_{\underline{j}}^{\rho\mu\nu\mathrm{sup}} := \sum_{e} \xi_{\sum_\alpha \alpha j_\alpha, e}^{\rho\mu\nu\mathrm{sup}}(g_{\underline{j}})t^e \tag{3.7}$$

is the *total character*.

3.2 No constraint on valences

In this section we consider the case $i = 0$, i.e. when all valences of vertices are allowed.

Proposition 3.1. *For all* $\rho, \mu, \nu \in \{+, -\}$

$$\begin{aligned}
\xi_{\underline{j}}^{\rho\mu\nu \not\circlearrowleft \not\rightleftharpoons} &= \prod_{\alpha\geq 1}\left(1 + t_{2\alpha-1}^{\mu}\right)^{(\alpha-1) j_{2\alpha-1}} \\
&\quad \prod_{\alpha\geq 1}\left(\rho\left(1 + \nu t_{\alpha}^{\mu}\right)\left(1 + t_{2\alpha}^{\mu}\right)^{\alpha-1}\right)^{j_{2\alpha}} \\
&\quad \prod_{\alpha\geq 1}(1 + t_{\alpha}^{\mu})^{\alpha \frac{j_\alpha(j_\alpha-1)}{2}} \\
&\quad \prod_{\beta>\alpha\geq 1}\left(1 + t_{\mathrm{lcm}(\alpha,\beta)}^{\mu}\right)^{\gcd(\alpha,\beta) j_\alpha j_\beta} \\
&= \prod_{\alpha\geq 1}\left(1 + t_{2\alpha-1}^{\mu}\right)^{-\frac{j_{2\alpha-1}}{2}} \\
&\quad \prod_{\alpha\geq 1}\left(\rho \frac{1 + \nu t_{\alpha}^{\mu}}{1 + t_{2\alpha}^{\mu}}\right)^{j_{2\alpha}} \\
&\quad \prod_{\beta,\alpha\geq 1}\left(1 + t_{\mathrm{lcm}(\alpha,\beta)}^{\mu}\right)^{\gcd(\alpha,\beta) \frac{j_\alpha j_\beta}{2}},
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\xi_{\underline{j}}^{\rho\mu+ \not\rightleftharpoons} &= \prod_{\alpha\geq 1}\left(1 + t_{2\alpha-1}^{\mu}\right)^{\alpha j_{2\alpha-1}} \\
&\quad \prod_{\alpha\geq 1}\left(\rho\left(1 + t_{\alpha}^{\mu}\right)\left(1 + t_{2\alpha}^{\mu}\right)^{\alpha}\right)^{j_{2\alpha}} \\
&\quad \prod_{\alpha\geq 1}(1 + t_{\alpha}^{\mu})^{\alpha \frac{j_\alpha(j_\alpha-1)}{2}} \\
&\quad \prod_{\beta>\alpha\geq 1}\left(1 + t_{\mathrm{lcm}(\alpha,\beta)}^{\mu}\right)^{\gcd(\alpha,\beta) j_\alpha j_\beta} \\
&= \xi_{\underline{j}}^{\rho\mu+ \not\circlearrowleft \not\rightleftharpoons} \prod_{\alpha\geq 1}\left(1 + t_{\alpha}^{\mu}\right)^{j_\alpha},
\end{aligned} \tag{3.9}$$

$$
\begin{aligned}
\xi_{\underline{j}}^{\rho+\nu\,\not\circlearrowleft} &= \prod_{\alpha\geq1}\left(\frac{1}{1-t^{+}_{2\alpha-1}}\right)^{(\alpha-1)j_{2\alpha-1}} \\
&\quad \prod_{\alpha\geq1}\left(\rho\frac{1}{1-\nu t^{+}_{\alpha}}\left(\frac{1}{1-t^{+}_{2\alpha}}\right)^{\alpha-1}\right)^{j_{2\alpha}} \\
&\quad \prod_{\alpha\geq1}\left(\frac{1}{1-t^{+}_{\alpha}}\right)^{\alpha\frac{j_\alpha(j_\alpha-1)}{2}} \\
&\quad \prod_{\beta>\alpha\geq1}\left(\frac{1}{1-t^{+}_{\mathrm{lcm}(\alpha,\beta)}}\right)^{\gcd(\alpha,\beta)j_\alpha j_\beta} \\
&= \prod_{\alpha\geq1}\left(1-t^{+}_{2\alpha-1}\right)^{\frac{j_{2\alpha-1}}{2}} \\
&\quad \prod_{\alpha\geq1}\left(\rho\frac{1-t^{+}_{2\alpha}}{1-\nu t^{+}_{\alpha}}\right)^{j_{2\alpha}} \\
&\quad \prod_{\beta,\alpha\geq1}\left(\frac{1}{1-t^{+}_{\mathrm{lcm}(\alpha,\beta)}}\right)^{\gcd(\alpha,\beta)\frac{j_\alpha j_\beta}{2}} ,
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
\xi_{\underline{j}}^{\rho++} &= \prod_{\alpha\geq1}\left(\frac{1}{1-t^{+}_{2\alpha-1}}\right)^{(\alpha-1)j_{2\alpha-1}} \\
&\quad \prod_{\alpha\geq1}\left(\rho\frac{1}{1-t^{+}_{\alpha}}\left(\frac{1}{1-t^{+}_{2\alpha}}\right)^{\alpha-1}\right)^{j_{2\alpha}} \\
&\quad \prod_{\alpha\geq1}\left(\frac{1}{1-t^{+}_{\alpha}}\right)^{\alpha\frac{j_\alpha(j_\alpha-1)}{2}} \\
&\quad \prod_{\beta>\alpha\geq1}\left(\frac{1}{1-t^{+}_{\mathrm{lcm}(\alpha,\beta)}}\right)^{\gcd(\alpha,\beta)j_\alpha j_\beta} \\
&= \xi_{\underline{j}}^{\rho++\,\not\circlearrowleft}\prod_{\alpha\geq1}\left(\frac{1}{1-t^{+}_{\alpha}}\right)^{j_\alpha} ,
\end{aligned}
\tag{3.11}
$$

where $t^{+}_{\alpha} := t^{\alpha}$ *and* $t^{-}_{\alpha} := -(-t)^{\alpha}$.

Proof. For the illustration we calculate the total character $\xi_{\underline{j}}^{\rho\mu+\,\not\Leftrightarrow}$. It is the polynomial in t with the coefficient of t^e being the character of $g_{\underline{j}} \in S_v$, $v := \sum_\alpha \alpha j_\alpha$, of the representation on

$$\mathrm{S} := \bar{\mathrm{V}}_v\mathrm{E}_e^{\mu+}\mathrm{H}_0\mathrm{GS}^{\not\Leftrightarrow} \otimes \mathrm{sgn}_v^{\rho} .$$

A basis of S consists of graphs with v distinguishable vertices and e unoriented indistinguishable edges. Sign part sgn_v^{ρ} is one-dimensional, so it does not effect the basis. An element $g_{\underline{j}} \in S_v$ acts on S by moving one graph to another. To calculate the character, we need to find graphs x moved to kx for a scalar k. By the definition of S one graph is moved to $\pm$ of another, so $k \in \{1, -1\}$. So we need to find graphs $x \in \bar{\mathrm{V}}_v\mathrm{E}_e^{\mu+}\mathrm{H}_0\mathrm{GS}^{\not\Leftrightarrow}$ that are not changed by the action of $g_{\underline{j}}$ and calculate the sign k that comes from the action on sgn_v^{ρ}.

The element $g_{\underline{j}}$ does not change the graph x if it has a symmetry of $g_{\underline{j}}$ on vertices, i. e. whose vertices are partitioned into the j_α cycles of α edges with circular symmetry in every cycle. The cycles are distinguishable, and the beginning vertex in the cycle is marked.

Let us pick a cycle with odd number $2\alpha-1$ of vertices, and let us number the vertices by $1, 2, \dots, 2\alpha-1$. If there is an edge between vertex 1 and l, because of the symmetry there should be also an edge between vertex 2 and $l+1$, and so on. We obtain $2\alpha-1$ edges in total. Graphs containing these edges contribute to the total character $\xi_{\underline{j}}^{\rho\mu+\,\not\Leftrightarrow}$ by multiplication with $t^{-}_{2\alpha-1} = -(-t)^{2\alpha-1} = t^{2\alpha-1}$. Note that cycling an odd number of edges is an even permutation, so it does not change sign. Graphs not containing these edges contribute to the total character by multiplication with 1, so this possibility contribute by $1 + t^{-}_{2\alpha-1}$ (recall that there are no multiple edges). There are α possibilities of putting that cycle of edges, so the contribution is $\left(1 + t^{-}_{2\alpha-1}\right)^{\alpha}$. The contributions of all j_α

cycles is $\left(1+t^{-}_{2\alpha-1}\right)^{\alpha j_{2\alpha-1}}$, and the contribution of all odd cycles is $\prod_{\alpha\geq 1}\left(1+t^{-}_{2\alpha-1}\right)^{\alpha j_{2\alpha-1}}$. This is the first line of the formula.

The second line is the contribution of even cycles. The third line is the contribution of the connections between two cycles of the same size, and the forth is the same for cycles of different sizes. The detailed derivation is straightforward and will be left to the reader. The similar calculation of the other total characters will also be left to the reader. □

Note that $\xi^{\rho\mu-}_{\underline{j}} = \xi^{\rho\mu-\not\circlearrowleft}_{\underline{j}}$, $\xi^{\rho-\nu}_{\underline{j}} = \xi^{\rho-\nu\not\leftrightarrow}_{\underline{j}}$ and $\xi^{\rho--}_{\underline{j}} = \xi^{\rho--\not\circlearrowleft\not\leftrightarrow}_{\underline{j}}$, so the proposition indeed covers all possibilities.

3.3 At least one-valent vertices

In this section we consider the case $i = 1$, i.e. when all vertices are at least one-valent. It means there are no isolated vertices.

Proposition 3.2. *For all* $\rho,\mu,\nu\in\{+,-\}$ *and* $\mathrm{sup}\in\{\emptyset,\not\circlearrowleft,\not\leftrightarrow,\not\circlearrowleft\not\leftrightarrow\}$

$$P^{\rho\mu\nu\geq 1\mathrm{sup}} = (1-\rho s)^{\rho}P^{\rho\mu\nu\mathrm{sup}}. \tag{3.12}$$

Proof. If $\rho=+$, adding a vertex to a graph leads to another graph which clearly does not have odd simetries, so it is not 0. In the other way, if a graph has an isolated vertex, deliting them leads to another graph. Therefore, the number of graphs with v vertices and e edges without isolated vertices is the number of graphs minus the number of graphs with at least 1 isolated vertex, that is the number of graphs with $v-1$ vertices and e edges, so

$$P^{+\mu\nu\geq 1\mathrm{sup}} = P^{+\mu\nu\mathrm{sup}}(1-s).$$

If $\rho=-$ there are no graphs with more than one isolated vertex, because interchanging two of them is an odd symmetry of the graph, implying that graph is zero. Therefore, by subtracting graphs with $v-1$ vertices we have also subtracted those with an isolated vertex, which do not lead to a graph by adding another isolated vertex. To fix this, we need to add graphs with $v-2$ vertices. The procedure will continue, so

$$P^{-\mu\nu\geq 1\mathrm{sup}} = P^{-\mu\nu\mathrm{sup}}(1-s+s^2-s^3+\dots) = P^{-\mu\nu\mathrm{sup}}\frac{1}{1+s}.$$

□

3.4 At least two-valent vertices

In this section we consider the case $i=2$, i.e. when all vertices are at least two-valent.

Proposition 3.3. *For all* $\rho,\mu,\nu\in\{+,-\}$ *and* $\mathrm{sup}\in\{\emptyset,\not\circlearrowleft,\not\leftrightarrow,\not\circlearrowleft\not\leftrightarrow\}$

$$P^{\rho\mu\nu\geq 2\mathrm{sup}} = (1-\rho s)^{\rho}\left(1-\mu s^2 t\right)^{\frac{\mu}{2}(\nu\rho-1)}\sum_{j_1,j_2,\dots\geq 0}\left(\prod_{\alpha\geq 1}\frac{s^{\alpha j_\alpha}}{j_\alpha!\alpha^{j_\alpha}}\right)P^{\rho\mu\nu\mathrm{sup}}\prod_{\alpha\geq 1}\left(1-(\mu\rho s t)^{\alpha}\right)^{\mu\rho j_\alpha}. \tag{3.13}$$

Proof. It holds that

$$P^{\rho\mu\nu\geq 2\mathrm{sup}} = \sum_{j_1,j_2,\dots\geq 0}\prod_{\alpha\geq 1}\frac{s^{\alpha j_\alpha}}{j_\alpha!\alpha^{j_\alpha}}\xi^{\nu\mu\rho}_{\underline{j},2} \tag{3.14}$$

where $\xi^{\rho\mu\nu\geq 2\mathrm{sup}}_{\underline{j}}$ is the total character for this case. It is constructed in the same way as $\xi^{\rho\mu\nu\mathrm{sup}}_{\underline{j}}$ from Proposition 3.1, by considering graphs with the symmetry of $g_{\underline{j}}$ on vertices, i. e. whose vertices are partitioned into the j_α cycles of α edges with circular symmetry in every cycle, but with ignoring addends which come from graphs without 0-valent or 1-valent vertices. Note that, because of the symmetry, the valences of vertices in the same cycle are the same, so we can talk about the valence of the cycle.

However, we can not directly force the cycles to be at least 2-valent. Instead of that we construct a *special character* $\overline{\xi}^{\rho\mu\nu\geq 2\mathrm{sup}}_{\underline{j}}$, which is constructed in the same way as a character, but we are allowed to put *special cycles* for which we know that they are 0 or 1-valent, together with edges. So, $\overline{\xi}^{\rho\mu\nu\geq 2\mathrm{sup}}_{\underline{j}}$ is the polynomial in variables t

and s^ρ_α for $\alpha \in \mathbb{N}$ where the coefficient next to $t^e \prod_\alpha \left(s^\rho_\alpha\right)^{n_\alpha}$ is the number of the graphs (counted with appropriate signs) with j_α distinguishable α-cycles, n_α indistinguishable special n_α-cycles and e edges. All cycles have a marked “first” vertex. If there is a symmetry of the order r between special cycles, we divide the term with r, in order not to count them multiple times, what will be clear later. We come to the following formula.

$$
\begin{aligned}
\overline{\xi}^{\rho\mu\nu\geq 2\mathrm{sup}}_{\underline{j}} &= \xi^{\rho\mu\nu\mathrm{sup}}_{\underline{j}} \\
&\prod_{\alpha\geq 1}\prod_{c\geq 1}\exp\left[j_\alpha \alpha s^\rho_{c\alpha} t^\mu_{c\alpha}\right] && {}^{\alpha}\bullet\!\!-\!\!\circ^{c\alpha} \\
&\prod_{c\geq 1}\exp\left[\nu s^\rho_{2c} t^\mu_c\right] && {}^{2c}\otimes \\
&\prod_{c\geq 1}\exp\left[\frac{1}{2}c\left(s^\rho_c\right)^2 t^\mu_c\right] && {}^{c}\circ\!\!-\!\!\circ^{c} \\
&\prod_{c\geq 1}\exp\left[s^\rho_c\right]. && {}^{c}\circ
\end{aligned}
\tag{3.15}
$$

The diagrams next to the factors depict the shape from which the factor comes. Full nodes ● represent general cycles in the graph, and empty nodes ○ represent special cycles added to the graph, which must be at most 1-valent. Small number next to a node $^{\alpha}$● represent the order (number of vertices) of the cycle. The symbol ⊗ represents a special even cycle with opposite vertices connected. A connection ○–○ or ●–○ represents a set of edges connecting vertices from different cycles respecting the symmetry, such that every vertex of the special cycle is adjacent to one edge.

The number of edges in a connection is therefore equal to the order of the special cycle. In particular, the existence of a connection between two special cycles implies that they have the same order, and the existence of a connection between a general cycle and a special cycle $^{\alpha}$●–○$^{\beta}$ implies $\alpha|\beta$. Note that cycles ⊗ have inner valence 1, and thus can not be connected to anything else. We can easily see that above diagrams represent all possible ways of adding special cycles to the graph.

The first factor is the contribution of graphs with no special cycles, known from Proposition 3.1. The second factor, a contribution of a special cycles connected to one general cycles we call *antennas*. Contributions of special-cycle structures disconnected from the rest of the graph, i.e. three last factors, we call *vacuum*. For the illustration of deriving the formula we explain the second factor, the antenna on an α-cycle.

To an α-cycle we can connect an antenna that is a special cycle whose order is a multiple of α. The contribution of all antennas is the product over all $\alpha \geq 1$ and all possibilities of orders of the special cycle $c\alpha$ for $c \geq 1$, of the contribution of such type of antennas, i.e. of antennas on α-cycle with the special cycle of order $c\alpha$.

Let ξ be the contribution of exactly 1 such antenna. There can be any number of such antennas on generally different α-cycles. If there are n of them on different cycles, the contribution is $\frac{1}{n!}\xi^n$ in order not to count same cases multiple times. Even if some of them are on the same cycle, because of the symmetry factor the contribution remains $\frac{1}{n!}\xi^n$. So, the total contribution of that type of antennas is $1 + \xi + \frac{1}{2}\xi^2 + \frac{1}{3!}\xi^3 + \cdots = \exp(\xi)$.

To calculate ξ we chose an α-cycle in j_α possible ways. We can connect the special cycle with the α-cycle in α different ways (recall that cycles have marked first vertex). We add a special cycle $s^\rho_{c\alpha}$ and a cycle of edges $t^\mu_{c\alpha} = \mu(\mu t)^{c\alpha}$. Multiplying everything leads to the ξ of the first factor. Other factors are similar. After simplification we have:

$$
\overline{\xi}^{\rho\mu\nu\geq 2\mathrm{sup}}_{\underline{j}} = (1-\rho s)^\rho \left(1-\mu s^2 t\right)^{\frac{\mu}{4}(\nu\rho-1)} \overline{\xi}^{\rho\mu\nu\mathrm{sup}}_{\underline{j}} \prod_{\alpha\geq 1}\left(1-(\mu\rho s t)^\alpha\right)^{\mu\rho j_\alpha}. \tag{3.16}
$$

To get the total character $\xi^{\rho\mu\nu\geq 2\mathrm{sup}}_{\underline{j}}$ we start with the total character $\xi^{\rho\mu\nu\mathrm{sup}}_{\underline{j}}$ of all-valence cycles, and subtract the character of graphs with the same cycles, of which one is special. We subtracted graphs with two special cycles twice, so we need to add the character of graphs with 2 special cycles. Than we need to subtract the character of graphs with 3 special cycles, add with 4, etc.

So all special characters $\overline{\xi}^{\rho\mu\nu\geq 2\mathrm{sup}}_{\underline{j}}$ for $\underline{k} \leq \underline{j}$ contribute to the total character $\xi^{\rho\mu\nu\geq 2\mathrm{sup}}_{\underline{j}}$, namely the coefficient (a polynomial in t) next to $\prod_\alpha \left(s^\rho_\alpha\right)^{j_\alpha - k_\alpha}$ with a sign $(-1)^{\sum_\alpha j_\alpha - k_\alpha}$. But all cycles in $\xi^{\rho\mu\nu\geq 2\mathrm{sup}}_{\underline{j}}$ are distinguishable while the special cycles contributing to $\overline{\xi}^{\rho\mu\nu\geq 2\mathrm{sup}}_{\underline{j}}$ are not. We can put $j_\alpha - k_\alpha$ special indistinguishable cycles between

k_α ordered cycles in $j_\alpha!/k_\alpha!$ ways. So the contribution of the coefficient next to $\prod_\alpha \left(s^\rho_\alpha\right)^{j_\alpha - k_\alpha}$ in $\bar{\xi}^{\rho\mu\nu\geq 2\mathrm{sup}}_{\underline{j}}$ into $\xi^{\rho\mu\nu\geq 2\mathrm{sup}}_{\underline{j}}$ is multiplied by $(-1)^{\sum_\alpha j_\alpha - k_\alpha} k_\alpha!/j_\alpha!$.

Therefore, if we put $s^+_c = \frac{-s^c}{c}$ and $s^-_c = \frac{(-s)^c}{c}$ we arrive at the formula:

$$\sum_{j_1, j_2, \cdots \geq 0} \prod_{\alpha \geq 1} \frac{s^{\alpha j_\alpha}}{j_\alpha! \alpha^{j_\alpha}} \bar{\xi}^{\rho\mu\nu\geq 2\mathrm{sup}}_{\underline{j}} = \sum_{j_1, j_2, \cdots \geq 0} \prod_{\alpha \geq 1} \frac{s^{\alpha j_\alpha}}{j_\alpha! \alpha^{j_\alpha}} \xi^{\rho\mu\nu\geq 2\mathrm{sup}}_{\underline{j}} = P^{\rho\mu\nu\geq 2\mathrm{sup}} \tag{3.17}$$

leading to the formula from the Proposition. □

Remark 3.4. Same procedure from the Proposition 3.3 will lead to the result of Proposition 3.2, although it is more complicated way. In the expression (3.15) only the last vacuum factor will remain, the one that is exactly equal to $(1 - \rho s)^\rho$.

3.5 At least three-valent vertices

In this section we consider the case $i = 3$, i.e. when all vertices are at least three-valent.

Proposition 3.5. *For all* $\rho, \mu, \nu \in \{+, -\}$ *and* $\mathrm{sup} \in \{\emptyset, \circlearrowleft, \nleftrightarrow, \circlearrowleft\nleftrightarrow\}$

$$P^{\rho\mu\nu\geq 3\mathrm{sup}} = V^{\rho\mu\nu\mathrm{sup}} \sum_{j_1, j_2, \cdots \geq 0} \prod_{\alpha \geq 1} \frac{s^{\alpha j_\alpha}}{j_\alpha! \alpha^{j_\alpha}} \xi^{\rho\mu\nu\mathrm{sup}}_{\underline{j}} C^{\rho\mu\nu\mathrm{sup}}_{\underline{j}} A^{\rho\mu\nu\mathrm{sup}}_{\underline{j}} \tag{3.18}$$

where

$$\begin{aligned} C^{\rho\mu\nu\circlearrowleft\nleftrightarrow}_{\underline{j}} A^{\rho\mu\nu\circlearrowleft\nleftrightarrow}_{\underline{j}} &= C^{\rho\mu\nu\nleftrightarrow}_{\underline{j}} A^{\rho\mu\nu\nleftrightarrow}_{\underline{j}} := \\ &:= \prod_{\alpha,\beta} \left[\left((\rho s)^{\mathrm{lcm}(\alpha,\beta)} t^{2\,\mathrm{lcm}(\alpha,\beta)}, (st)^{2\,\mathrm{lcm}(\alpha,\beta)}\right)^\rho_\infty \left(s^{2\,\mathrm{lcm}(\alpha,\beta)} (\mu t)^{3\,\mathrm{lcm}(\alpha,\beta)}, (st)^{2\,\mathrm{lcm}(\alpha,\beta)}\right)^{-\mu}_\infty \right]^{\gcd(\alpha,\beta) j_\alpha j_\beta/2} \\ &\prod_\alpha \left[\left(1 - (\rho s)^\alpha t^{2\alpha}\right)^{-\rho(\mu+1)/2} \left((\mu\rho st)^\alpha, (st)^{2\alpha}\right)^{\mu\rho}_\infty \left((st)^{2\alpha}, (st)^{2\alpha}\right)^{-1}_\infty \right]^{j_\alpha} \\ &\left[\left(\rho s^{2\alpha-1} t^{4\alpha-2}, (st)^{8\alpha-4}\right)^{\mu\rho}_\infty \left(\mu s^{4\alpha-2} t^{6\alpha-3}, (st)^{8\alpha-4}\right)^{\nu\rho}_\infty \left(\rho s^{6\alpha-3} t^{8\alpha-4}, (st)^{8\alpha-4}\right)^{-1}_\infty \left(\mu s^{8\alpha-4} t^{10\alpha-5}, (st)^{8\alpha-4}\right)^{-\nu\mu}_\infty \right]^{j_{2\alpha-1}/2} \\ &\left[\left((\rho s)^\alpha t^{2\alpha}, (st)^{4\alpha}\right)^{\mu\rho}_\infty \left(s^{2\alpha} (\mu t)^{3\alpha}, (st)^{4\alpha}\right)^{\nu\rho}_\infty \left((\rho s)^{3\alpha} t^{4\alpha}, (st)^{4\alpha}\right)^{-1}_\infty \left(s^{4\alpha} (\mu t)^{5\alpha}, (st)^{4\alpha}\right)^{-\nu\mu}_\infty \right]^{j_{2\alpha}}, \end{aligned} \tag{3.19}$$

$$C^{\rho+\nu\circlearrowleft}_{\underline{j}} A^{\rho+\nu\circlearrowleft}_{\underline{j}} = C^{\rho+\nu}_{\underline{j}} A^{\rho+\nu}_{\underline{j}} := C^{\rho+\nu\circlearrowleft\nleftrightarrow}_{\underline{j}} A^{\rho+\nu\circlearrowleft\nleftrightarrow}_{\underline{j}} \sum_\alpha \left(1 - (\rho s)^\alpha t^{2\alpha}\right)^{\rho j_\alpha}, \tag{3.20}$$

$$\begin{aligned} V^{\rho\mu\nu\circlearrowleft\nleftrightarrow} &:= \left(\rho s, (st)^4\right)^\rho_\infty \left(\mu s^2 t, (st)^4\right)^{\mu(\nu\rho-1)/2}_\infty \left(\rho s^3 t^2, (st)^4\right)^{(\rho-\mu)/2}_\infty \left(\mu s^4 t^3, (st)^4\right)^{-(\mu+\nu)/2}_\infty \\ &(1 - \mu\rho st)^{-\mu\rho/2} \left(1 - (st)^2\right)^{(1-\mu\rho)/4} (-st, \mu\rho st)^{-1/2}_\infty \left((st)^4, (st)^2\right)^{(\rho-\mu)/4}_\infty \left(\mu\rho (st)^5, (st)^4\right)^{\nu\mu\rho/2}_\infty \left(\mu\rho (st)^3, (st)^4\right)^{-\nu/2}_\infty, \end{aligned} \tag{3.21}$$

$$V^{\rho\mu+\nleftrightarrow} := V^{\rho\mu+\circlearrowleft\nleftrightarrow} (1 - \mu\rho st)^{\mu\rho}, \tag{3.22}$$

$$V^{\rho+\nu\circlearrowleft} := V^{\rho+\nu\circlearrowleft\nleftrightarrow} \left(1 - (st)^2\right)^{(\rho-1)/2}, \tag{3.23}$$

$$V^{\rho++} := V^{\rho++\circlearrowleft\nleftrightarrow} (1 - \rho st)^\rho \left(1 - (st)^2\right)^{(\rho-1)/2}. \tag{3.24}$$

and $\xi^{\rho\mu\nu\mathrm{sup}}_{\underline{j}}$ *is the character from Proposition 3.1, where*

$$(a, q)_\infty := \prod_{k \geq 0} \left(1 - aq^k\right) \tag{3.25}$$

is the q-Pochhammer symbol.

Proof. We proceed in the same way as in Proposition 3.3 and use the formula

$$P^{\rho\mu\nu\geq 3\mathrm{sup}} = \sum_{j_1,j_2,\dots\geq 0} \prod_{\alpha\geq 1} \frac{s^{\alpha j_\alpha}}{j_\alpha!\alpha^{j_\alpha}} \overline{\xi}^{\rho\mu\nu\geq 3\mathrm{sup}}_{\underline{j}} \tag{3.26}$$

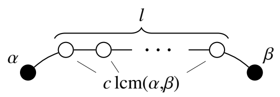
where $\overline{\xi}^{\rho\mu\nu\geq 3\mathrm{sup}}_{\underline{j}}$ is the special character for this case, so special cycles are now allowed to be 2-valent. Special cycles can still be added to the general cycles of partition $\underline{j}$ in a controlled way: they are either disconnected from the rest and form free loops or lines (vacuum), connected to one cycle (antennas) or connect two cycles (connections). So

$$\overline{\xi}^{\rho\mu\nu\geq 3\mathrm{sup}}_{\underline{j}} = \xi^{\rho\mu\nu\mathrm{sup}}_{\underline{j}} C^{\rho\mu\nu\mathrm{sup}}_{\underline{j}} A^{\rho\mu\nu\mathrm{sup}}_{\underline{j}} V^{\rho\mu\nu\mathrm{sup}}_{\underline{j}} \tag{3.27}$$

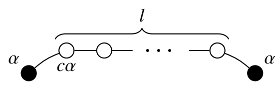
where $C^{\rho\mu\nu\mathrm{sup}}_{\underline{j}}$, $A^{\rho\mu\nu\mathrm{sup}}_{\underline{j}}$, and $V^{\rho\mu\nu\mathrm{sup}}_{\underline{j}}$ are connection, antenna and vacuum factor respectively. Careful calculation leads to the following formulas for them.

(3.28)

$$C^{\rho\mu\nu\mathrm{sup}}_{\underline{j}} = \prod_{\beta>\alpha\geq 1} \prod_{c\geq 1} \prod_{l\geq 1} \exp\left[j_\alpha j_\beta \alpha\beta (c\,\mathrm{lcm}(\alpha,\beta))^{l-1} \left(s^{\rho}_{c\,\mathrm{lcm}(\alpha,\beta)}\right)^{l} \left(t^{\mu}_{c\,\mathrm{lcm}(\alpha,\beta)}\right)^{l+1} \right]$$



$$\prod_{\alpha\geq 1} \prod_{c\geq 1} \prod_{l\geq 1} \exp\left[\frac{j_\alpha(j_\alpha-1)}{2} \alpha^2 (c\alpha)^{l-1} \left(s^{\rho}_{c\alpha}\right)^{l} \left(t^{\mu}_{c\alpha}\right)^{l+1} \right]$$



The construction is the same as one in Proposition 3.3. For the illustration we explain the first factor, the contribution of connections between cycles of different order, say an α-cycle and a β-cycle, $\beta < \alpha$. The two cycles can be connected via a chain of l special cycles. Because all special cycles are connected to two cycles, there can not be internal connections in the cycles and from each vertex exactly 1 edge goes to the next and to the previous cycle. Because of connecting rules, the order of all special cycles in the chain is the same and it is a multiple of the least common multiple $\mathrm{lcm}(\alpha,\beta)$. So, the contribution of all connections between different cycles is the product over all $\beta > \alpha \geq 1$, all chain lengths $l \geq 1$ and all possibilities of orders of special cycles $c\,\mathrm{lcm}(\alpha,\beta)$ for $c \geq 1$, of the contribution of such type of connections, i.e. of connections between α and β-cycle of length l and order of special cycle $c\,\mathrm{lcm}(\alpha,\beta)$.

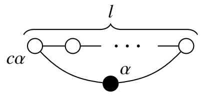
Let ξ be the contribution of exactly 1 such connection. With the same argument as before we conclude that the total contribution of this type of connection is $\exp(\xi)$. To calculate ξ we first chose an α-cycle and β-cycle in $j_\alpha j_\beta$ possible ways. We can connect the first special cycle with the α-cycle in α different ways, and the last one with the β-cycle in β different ways. Connections between special cycles can be done in $(c\,\mathrm{lcm}(\alpha,\beta))^{l-1}$ different ways. We also add l special cycles $\left(s^{\rho}_{c\,\mathrm{lcm}(\alpha,\beta)}\right)^{l}$ and $l+1$ cycles of edges $\left(t^{\mu}_{c\,\mathrm{lcm}(\alpha,\beta)}\right)^{l+1}$. Multiplying everything leads to the ξ of the first factor. Second factor is similar. Note that, for the connections, tadpoles and multiple edges are not possible, so the formula does not change if we allow them or not. It also does not depend on ν.

The antenna parts are:

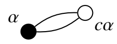
(3.29)

$$A_{\underline{j}}^{\rho\mu\nu\,\circ\,\not\multimap} = A_{\underline{j}}^{\rho\mu\nu\not\multimap} =$$

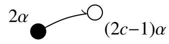
$$\prod_{\alpha\geq 1}\prod_{c\geq 1}\prod_{l\geq 2}\exp\left[\frac{1}{2}j_\alpha\alpha^2(c\alpha)^{l-1}\left(s^\rho_{c\alpha}\right)^l\left(t^\mu_{c\alpha}\right)^{l+1}\right]$$



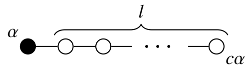
$$\prod_{\alpha\geq 1}\prod_{c\geq 1}\exp\left[j_\alpha\frac{\alpha(\alpha-1)}{2}s^\rho_{c\alpha}\left(t^\mu_{c\alpha}\right)^2\right]$$



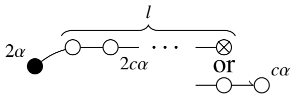
$$\prod_{\alpha\geq 1}\prod_{c\geq 1}\exp\left[j_{2\alpha}\alpha\left(s^\rho_{(2c-1)\alpha}\right)\left(t^\mu_{2(2c-1)\alpha}\right)\right]$$



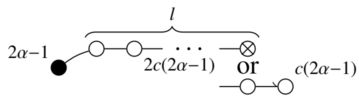
$$\prod_{\alpha\geq 1}\prod_{c\geq 1}\prod_{l\geq 1}\exp\left[j_\alpha\alpha(c\alpha)^{l-1}\left(s^\rho_{c\alpha}\right)^l\left(t^\mu_{c\alpha}\right)^l\right]$$



$$\prod_{\alpha\geq 1}\prod_{c\geq 1}\prod_{l\geq 1}\exp\left[j_{2\alpha}(2\alpha)^l(c)^{l-1}\left(s^\rho_{2c\alpha}\right)^l\left(t^\mu_{2c\alpha}\right)^l\left(\nu t^\mu_{c\alpha}+c\alpha s^\rho_{c\alpha}t^\mu_{2c\alpha}\right)\right]$$

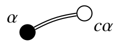


$$\prod_{\alpha\geq 1}\prod_{c\geq 1}\prod_{l\geq 1}\exp\left[j_{2\alpha-1}(2\alpha-1)^l(2c)^{l-1}\left(s^\rho_{2c(2\alpha-1)}\right)^l\left(t^\mu_{2c(2\alpha-1)}\right)^l\left(\nu t^\mu_{c(2\alpha-1)}+c(2\alpha-1)s^\rho_{c(2\alpha-1)}t^\mu_{2c(2\alpha-1)}\right)\right]$$



,

(3.30) $$A_{\underline{j}}^{\rho+\nu\,\circ} = A_{\underline{j}}^{\rho+\nu} = A_{\underline{j}}^{\rho+\nu\,\circ\,\not\multimap}\prod_{\alpha\geq 1}\prod_{c\geq 1}\exp\left[j_\alpha\alpha s^\rho_{c\alpha}\left(t^+_{c\alpha}\right)^2\right]$$

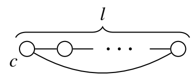


.

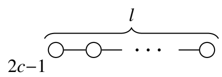
The vacuum parts are:

(3.31)

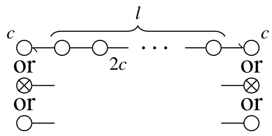
$$V_{\underline{j}}^{\rho\mu\nu\,\circ\,\not\multimap} = \prod_{c\geq 1}\prod_{l\geq 3}\exp\left[\frac{1}{2l}c^l\left(s^\rho_c\right)^l\left(t^\mu_c\right)^l\right]$$



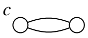
$$\prod_{c\geq 1}\prod_{l\geq 2}\exp\left[\frac{1}{2}(2c-1)^{l-1}\left(s^\rho_{2c-1}\right)^l\left(t^\mu_{2c-1}\right)^{l-1}\right]$$

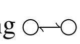


$$\prod_{c\geq 1}\prod_{l\geq 0}\exp\left[\frac{1}{2}(2c)^{l-1}\left(s^\rho_{2c}\right)^l\left(t^\mu_{2c}\right)^{l+1}\left(2cs^\rho_{2c}+\nu 2cs^\rho_{2c}t^\mu_c+cs^\rho_c\right)^2\right]$$



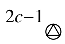
$$\prod_{c\geq 1}\exp\left[\frac{1}{2}\frac{c(c-1)}{2}\left(s^\rho_c\right)^2\left(t^\mu_c\right)^2-\frac{c}{4}(s^\rho_c)^2t^\mu_{2c}\right]$$



and cancelling  from above

$$\prod_{c\geq 1}\exp\left[(c-1)s^\rho_{2c}t^\mu_{2c}\right]$$

$$\prod_{c\geq 1}\exp\left[(c-1)s^\rho_{2c-1}t^\mu_{2c-1}\right]$$



$$\prod_{c\geq 1}\exp\left[\nu s^\rho_{2c}t^\mu_c\right]$$

$$\prod_{c\geq 1}\exp\left[s^\rho_c\right]$$

(3.32) $$V_{\underline{j}}^{\rho\mu+\not\multimap} = V_{\underline{j}}^{\rho\mu+\,\circ\,\not\multimap}\prod_{c\geq 1}\exp\left[s^\rho_ct^\mu_c\right]$$

$$
\begin{aligned}
V_{\underline{j}}^{\rho+\nu\,\varnothing} &= V_{\underline{j}}^{\rho+\nu\,\varnothing\,\nleftrightarrow} \prod_{c\geq 1} \exp\left[\frac{1}{2}c\left(s_c^{\rho}\right)^2 (t_c^+)^2\right] & {}^{c}\bigcirc\!=\!=\!\bigcirc^{c} \\
&\prod_{c\geq 1} \exp\left[s_{2c}^{\rho}\,(t_c^+)^2\right] & {}^{2c}\otimes
\end{aligned}
\tag{3.33}
$$

$$
V_{\underline{j}}^{\rho++} = V_{\underline{j}}^{\rho++\,\varnothing\,\nleftrightarrow} \prod_{c\geq 1} \exp\left[s_c^{\rho} t_c^+\right] \prod_{c\geq 1} \exp\left[\frac{1}{2}c\left(s_c^{\rho}\right)^2 (t_c^+)^2\right] \prod_{c\geq 1} \exp\left[s_{2c}^{\rho}\,(t_c^+)^2\right]
\tag{3.34}
$$

There are some new symbols in diagrams: $\circledast$ represents a special 2-valent cycle with a set of inside edges (not towards the opposite vertex, and not tadpoles) respecting the symmetry, $\bigcirc\!^{\circ}$ represent a special 2-valent cycle with tadpoles on each vertex, and $\otimes$ represent a special 2-valent cycle of even order with double edges towards the opposite vertex. "Harpooned" connection $\bigcirc\!\rightharpoonup\!\bigcirc$ or $\bullet\!\rightharpoonup\!\bigcirc$ represents a set of edges from a left-hand cycle of even order 2α to the right-hand cycle whose order is odd multiple of α, and thus every vertex of the right-hand cycle is by symmetry forced to be connected to two opposite vertices of the left-hand even cycle. Double connection $\bigcirc\!=\!\bigcirc$ or $\bullet\!=\!\bigcirc$ is similar to the single connection, but all edges are doubled.

Note that cycles $\otimes$ and $\circledast$ have inner valence 2. The existence of a double connection towards a special cycle ${}^{\alpha}\bullet\!=\!\bigcirc^{\beta}$ implies that $\alpha|\beta$ like the single cycle, and increases the valence of the special cycle by 2. Therefore a double connection between two special cycles implies that they have the same order. A "harpooned" connection towards a special cycle ${}^{2\alpha}\bullet\!\rightharpoonup\!\bigcirc^{\beta}$ implies that β is an odd multiple of α and increases the valence of the special cycle by two. If the other cycle is also special, its valence is increased by one, and it is $\alpha = \beta$.

The formulas from the proposition follow dirrectly after simplification. □

We will be particularly interested in the special cases of the above formula that appear in graph complexes, c.f. Definition 2.5:

$$
\begin{aligned}
P^{+-+\geq 3}(s,t) &= \frac{\left(s,(st)^2\right)_\infty}{(-st,(st)^2)_\infty} \sum_{j_1,j_2,\dots\geq 0} \prod_{\alpha} \frac{s^{\alpha j_\alpha}}{j_\alpha!\alpha^{j_\alpha}} \frac{1}{((-st)^\alpha,(-st)^\alpha)_\infty^{j_\alpha}} \\
&\left(\frac{\left((-t)^{2\alpha-1},(st)^{4\alpha-2}\right)_\infty}{(s^{2\alpha-1}t^{4\alpha-2},(st)^{4\alpha-2})_\infty}\right)^{j_{2\alpha-1}/2} \left(\frac{\left((-t)^{\alpha},(st)^{2\alpha}\right)_\infty}{(s^{\alpha}t^{2\alpha},(st)^{2\alpha})_\infty}\right)^{j_{2\alpha}} \prod_{\alpha,\beta} \left((-t)^{\mathrm{lcm}(\alpha,\beta)},(-st)^{\mathrm{lcm}(\alpha,\beta)}\right)_\infty^{\gcd(\alpha,\beta) j_\alpha j_\beta/2}.
\end{aligned}
\tag{3.35}
$$

$$
\begin{aligned}
P^{-+-\geq 3}(s,t) &= \frac{1}{(-s,(st)^2)_\infty\,((st)^2,(st)^2)_\infty} \sum_{j_1,j_2,\dots\geq 0} \prod_{\alpha} \frac{s^{\alpha j_\alpha}}{j_\alpha!\alpha^{j_\alpha}} \frac{1}{((-st)^\alpha,(-st)^\alpha)_\infty^{j_\alpha}} \\
&\left(\frac{\left(t^{2\alpha-1},(st)^{4\alpha-2}\right)_\infty}{((-s)^{2\alpha-1}t^{4\alpha-2},(st)^{4\alpha-2})_\infty}\right)^{j_{2\alpha-1}/2} \left(\frac{-\left(t^{\alpha},(st)^{2\alpha}\right)_\infty}{((-s)^{\alpha}t^{2\alpha},(st)^{2\alpha})_\infty}\right)^{j_{2\alpha}} \prod_{\alpha,\beta} \frac{1}{(t^{\mathrm{lcm}(\alpha,\beta)},(-st)^{\mathrm{lcm}(\alpha,\beta)})_\infty^{\gcd(\alpha,\beta) j_\alpha j_\beta/2}}.
\end{aligned}
\tag{3.36}
$$

$$
\begin{aligned}
P^{-+-\geq 3\,\nleftrightarrow}(s,t) &= \frac{1}{(-s,(st)^2)_\infty\,((st)^4,(st)^2)_\infty} \sum_{j_1,j_2,\dots\geq 0} \prod_{\alpha} \frac{s^{\alpha j_\alpha}}{j_\alpha!\alpha^{j_\alpha}} \left(\frac{1+\left(st^2\right)^\alpha}{((-st)^\alpha,(-st)^\alpha)_\infty}\right)^{j_\alpha} \\
&\left(\frac{\left(t^{2\alpha-1},(st)^{4\alpha-2}\right)_\infty}{(1-t^{4\alpha-2})\,((-s)^{2\alpha-1}t^{4\alpha-2},(st)^{4\alpha-2})_\infty}\right)^{j_{2\alpha-1}/2} \left(\frac{-\left(t^{\alpha},(st)^{2\alpha}\right)_\infty}{(1+t^{2\alpha})\,((-s)^{\alpha}t^{2\alpha},(st)^{2\alpha})_\infty}\right)^{j_{2\alpha}} \\
&\prod_{\alpha,\beta} \left(\frac{1-t^{2\,\mathrm{lcm}(\alpha,\beta)}}{(t^{\mathrm{lcm}(\alpha,\beta)},(-st)^{\mathrm{lcm}(\alpha,\beta)})_\infty}\right)^{\gcd(\alpha,\beta) j_\alpha j_\beta/2},
\end{aligned}
\tag{3.37}
$$

$$
\begin{aligned}
P^{+-+\geq 3\,\varnothing}(s,t) &= \frac{\left(s,(st)^2\right)_\infty}{(-(st)^3,(st)^2)_\infty} \sum_{j_1,j_2,\dots\geq 0} \prod_{\alpha} \frac{s^{\alpha j_\alpha}}{j_\alpha!\alpha^{j_\alpha}} \frac{1}{((-st)^\alpha,(-st)^\alpha)_\infty^{j_\alpha}} \\
&\left(\frac{\left(s^{4\alpha-2}(-t)^{6\alpha-3},(st)^{4\alpha-2}\right)_\infty}{(1+t^{2\alpha-1})\,(s^{2\alpha-1}t^{4\alpha-2},(st)^{4\alpha-2})_\infty}\right)^{j_{2\alpha-1}/2} \left(\frac{\left(s^{2\alpha}(-t)^{3\alpha},(st)^{2\alpha}\right)_\infty}{(1+(-t)^{\alpha})\,(s^{\alpha}t^{2\alpha},(st)^{2\alpha})_\infty}\right)^{j_{2\alpha}} \\
&\prod_{\alpha,\beta} \left((-t)^{\mathrm{lcm}(\alpha,\beta)},(-st)^{\mathrm{lcm}(\alpha,\beta)}\right)_\infty^{\gcd(\alpha,\beta) j_\alpha j_\beta/2}.
\end{aligned}
\tag{3.38}
$$

3.6 Euler characteristics

Calculation of dimensions of graph spaces can be used to calculate Euler characteristics of graph complexes. For Euler characteristics to be well defined the complex has to be finite-dimensional in each degree, and it has to be bounded. Tho whole complexes $\mathrm{fGC}_n^{\mathrm{sup}}$ are never finite-dimensional in each degree, but the subcomplexes

$$\mathrm{B}_b\mathrm{fGC}_n^{\mathrm{sup}}$$

are. Only the additional constraint ≥ 3 forces them to be bounded. So, we are interested in Euler characteristics

$$\chi\left(\mathrm{B}_b\mathrm{fGC}_n^{\geq 3\mathrm{sup}}\right) = \sum_{d\in\mathbb{Z}}(-1)^d \dim\left(\mathrm{D}_d\mathrm{B}_b\mathrm{fGC}_n^{\geq 3\mathrm{sup}}\right) = \sum_{v\geq 1}(-1)^{v-n-nb+b}\dim\left(\mathrm{V}_v\mathrm{E}_{v+b}\mathrm{fGC}_n^{\geq 3\mathrm{sup}}\right) = \tag{3.39}$$

$$= \sum_{v\geq 1}\left\{\begin{array}{ll} (-1)^{v+b}\dim_{v,v+b}^{+-+\geq 3\mathrm{sup}} & \text{for } n \text{ even,} \\ -(-1)^{v}\dim_{v,v+b}^{-+-\geq 3\mathrm{sup}} & \text{for } n \text{ odd} \end{array}\right. =: \left\{\begin{array}{ll} \chi_{b,even}^{\mathrm{sup}} & \text{for } n \text{ even,} \\ \chi_{b,odd}^{\mathrm{sup}} & \text{for } n \text{ odd,} \end{array}\right.$$

for sup $\in \{\emptyset, \not\circlearrowleft, \not\Leftrightarrow, \not\circlearrowleft\not\Leftrightarrow\}$.

3.7 The connected graphs

In this section we study dimensions of graphs spaces spanned by connected graphs. Let

$$\overline{\dim}_{v,e}^{\rho\mu\nu\mathrm{sup}} := \dim\left(\mathrm{V}_v^{\rho}\mathrm{E}_e^{\mu\nu}\mathrm{C}_1\mathrm{H}_0\mathrm{GS}^{\mathrm{sup}}\right) \tag{3.40}$$

Graphs (basis elements) in $\mathrm{V}_v^{\rho}\mathrm{E}_e^{\mu\nu}\mathrm{H}_0\mathrm{GS}^{\mathrm{sup}}$ are possibly disconnected. Let one of them consist of $j_{w,f}$ connected graphs in $\mathrm{V}_w^{\rho}\mathrm{E}_f^{\mu\nu}\mathrm{C}_1\mathrm{H}_0\mathrm{GS}^{\mathrm{sup}}$ for $w = 1, \ldots, v$ and $f = 0, \ldots, e$. It holds that

$$\sum_{w,f} w j_{w,f} = v, \quad \sum_{w,f} f j_{w,f} = e.$$

So we are choosing $j_{w,f}$ elements out of $\overline{\dim}_{w,f}^{\rho\mu\nu\mathrm{sup}}$ basis elements of $\mathrm{V}_v^{\rho}\mathrm{E}_e^{\mu\nu}\mathrm{C}_1\mathrm{H}_0\mathrm{GS}^{\mathrm{sup}}$.

This can be done in $\binom{\overline{\dim}_{w,f}^{\rho\mu\nu\mathrm{sup}}}{j_{w,f}}$ ways if the symmetry does not allow same connected components and in $(-1)^{j_{w,f}}\binom{-\overline{\dim}_{w,f}^{\rho\mu\nu\mathrm{sup}}}{j_{w,f}}$ if the symmetry does allow same connected components. The symmetry is $\rho^w\mu^f$, so this can generally be done in

$$\left(-\rho^w\mu^f\right)^{j_{w,f}}\binom{-\rho^w\mu^f\overline{\dim}_{w,f}^{\rho\mu\nu\mathrm{sup}}}{j_{w,f}} \tag{3.41}$$

ways. Therefore

$$\dim_{v,e}^{\rho\mu\nu\mathrm{sup}} = \sum_{\substack{(j_{w,f}\geq 0|w\geq 1, f\geq 0) \\ \sum_{w,f} w j_{w,f}=v \\ \sum_{w,f} f j_{w,f}=e}} \prod_{w,f}\left(-\rho^w\mu^f\right)^{j_{w,f}}\binom{-\rho^w\mu^f\overline{\dim}_{w,f}^{\rho\mu\nu\mathrm{sup}}}{j_{w,f}}. \tag{3.42}$$

We can also calculate the Euler characteristics. Let

$$\chi\left(\mathrm{B}_b\mathrm{C}_1\mathrm{fGC}_n^{\geq 3\mathrm{sup}}\right) =: \left\{\begin{array}{ll} \overline{\chi}_{b,even}^{\mathrm{sup}} & \text{for } n \text{ even,} \\ \overline{\chi}_{b,odd}^{\mathrm{sup}} & \text{for } n \text{ odd.} \end{array}\right. \tag{3.43}$$

For n odd it holds that

$$
\begin{aligned}
\chi^{\mathrm{sup}}_{b,odd} &= \sum_{v\geq 0} -(-1)^v \dim^{-+-\geq 3\mathrm{sup}}_{v,v+b} = \\
&= -\sum_{v\geq 0}(-1)^v \sum_{\substack{(j_{w,f}\geq 0 | w\geq 1, f\geq 0)\\ \sum_{w,f} w j_{w,f}=v \\ \sum_{w,f} f j_{w,f}=v+b}} \prod_{w,f} (-(-1)^w)^{j_{w,f}} \binom{-(-1)^w \overline{\dim}^{-+-\geq 3\mathrm{sup}}_{w,f}}{j_{w,f}} = \\
&= -\sum_{\substack{(j_{w,f}\geq 0 | w\geq 1, f\geq 0)\\ \sum_{w,f}(f-w) j_{w,f}=b}} (-1)^{\sum_{w,f} w j_{w,f}} \prod_{w,f} (-(-1)^w)^{j_{w,f}} \binom{-(-1)^w \overline{\dim}^{-+-\geq 3\mathrm{sup}}_{w,f}}{j_{w,f}} = \\
&= -\sum_{\substack{(j_{w,f}\geq 0 | w\geq 1, f\geq 0)\\ \sum_{w,f}(f-w) j_{w,f}=b}} \prod_{w,f} (-1)^{j_{w,f}} \binom{-(-1)^w \overline{\dim}^{-+-\geq 3\mathrm{sup}}_{w,f}}{j_{w,f}} = \\
&= -\sum_{\substack{(i_c\geq 0|c\in\mathbb{Z})\\ \sum_c c i_c=b}} \sum_{\substack{(j_{w,f}\geq 0 | w\geq 1, f\geq 0)\\ \sum_w j_{w,w+c}=i_c}} \prod_c \prod_w (-1)^{j_{w,w+c}} \binom{-(-1)^w \overline{\dim}^{-+-\geq 3\mathrm{sup}}_{w,w+c}}{j_{w,w+c}} = \\
&= -\sum_{\substack{(i_c\geq 0|c\in\mathbb{Z})\\ \sum_c c i_c=b}} \prod_c (-1)^{i_c} \sum_{\substack{(j_w\geq 0|w\geq 1)\\ \sum_w j_w=i_c}} \prod_w \binom{-(-1)^w \overline{\dim}^{-+-\geq 3\mathrm{sup}}_{w,w+c}}{j_w}.
\end{aligned}
\tag{3.44}
$$

Lemma 3.6. *For every $i\in\mathbb{N}$ it holds that*

$$
\sum_{\substack{(j_w\geq 0|w\geq 1)\\ \sum_w j_w=i}} \prod_w \binom{-(-1)^w \overline{\dim}^{-+-\geq 3\mathrm{sup}}_{w,w+c}}{j_w} = \binom{\overline{\chi}^{\mathrm{sup}}_{b,odd}}{i}.
$$

Proof. Let x be a formal variable.

$$
\begin{aligned}
&\sum_i x^i \sum_{\substack{(j_w\geq 0|w\geq 1)\\ \sum_w j_w=i}} \prod_w \binom{-(-1)^w \overline{\dim}^{-+-\geq 3\mathrm{sup}}_{w,w+c}}{j_w} = \sum_i \sum_{\substack{(j_w\geq 0|w\geq 1)\\ \sum_w j_w=i}} \prod_w \binom{-(-1)^w \overline{\dim}^{-+-\geq 3\mathrm{sup}}_{w,w+c}}{j_w} x^{j_w} = \\
&= \sum_{(j_w\geq 0|w\geq 1)} \prod_w \binom{-(-1)^w \overline{\dim}^{-+-\geq 3\mathrm{sup}}_{w,w+c}}{j_w} x^{j_w} = \prod_w \sum_j \binom{-(-1)^w \overline{\dim}^{-+-\geq 3\mathrm{sup}}_{w,w+c}}{j} x^j = \\
&\prod_w (1+x)^{-(-1)^w \overline{\dim}^{-+-\geq 3\mathrm{sup}}_{w,w+c}} = (1+x)^{-\sum_w (-1)^w \overline{\dim}^{-+-\geq 3\mathrm{sup}}_{w,w+c}} = (1+x)^{\chi\left(\mathrm{B}_b\mathrm{C}_1\mathrm{fGC}^{\geq 3\mathrm{sup}}_n\right)} = \sum_i \binom{\chi\left(\mathrm{B}_b\mathrm{C}_1\mathrm{fGC}^{\geq 3\mathrm{sup}}_n\right)}{i} x^i
\end{aligned}
$$

□

The lemma implies that

$$
\chi^{\mathrm{sup}}_{b,odd} = -\sum_{\substack{(i_c\geq 0|c\in\mathbb{Z})\\ \sum_c c i_c=b}} \prod_c (-1)^{i_c} \binom{\overline{\chi}^{\mathrm{sup}}_{b,odd}}{i_c}.
\tag{3.45}
$$

A similar argument leads to the similar formula for n even:

$$
\chi^{\mathrm{sup}}_{b,even} = \sum_{\substack{(i_c\geq 0|c\in\mathbb{Z})\\ \sum_c c i_c=b}} \prod_c (-1)^{i_c} \binom{-\overline{\chi}^{\mathrm{sup}}_{b,even}}{i_c}.
\tag{3.46}
$$

Formulas (3.42), (3.45) and (3.46) calculate dimensions and Euler characteristics of whole spaces and complexes from the data about connected spaces and complexes. Indeed, in previous sections we calculated dimensions and Euler characteristics for whole complexes, and we are interested in the same data in connected case. Therefore, the formulas are used recursively to calculate the connected case from the general one.

3.8 Numerical data

The formulas from this chapter can be used to calculate the dimensions of the spaces $\mathrm{V}_v^{\rho}\mathrm{E}_e^{\mu\nu}\mathrm{H}_0\mathrm{GS}^{\mathrm{sup}}$ and $\mathrm{V}_v^{\rho}\mathrm{E}_e^{\mu\nu}\mathrm{C}_1\mathrm{H}_0\mathrm{GS}^{\mathrm{sup}}$ using the computer. An example of the computer program is written in Appendix C. In Table 3.1 we list the dimensions $\overline{\dim}_{v,e}^{+-+\geq 3\,\not\circ}$ and $\overline{\dim}_{v,e}^{-+-\geq 3\,\not\parallel}$ for v up to 24 and e up to 36, modulo the product of prime numbers $3999971 \cdot 3999949 \cdot 3999929 \cdot 3999923 \cdot 3999917 \cdot 3999901 \cdot 3999893 \cdot 3999971$.

Our results can be used to calculate the Euler characteristics of the graph complexes $\chi_{b,even}$, $\chi_{b,odd}$, $\chi_{b,even}^{\not\circ}$ and $\chi_{b,odd}^{\not\parallel}$, as well as for connected case $\overline{\chi}_{b,even}$, $\overline{\chi}_{b,odd}$, $\overline{\chi}_{b,even}^{\not\circ}$ and $\overline{\chi}_{b,odd}^{\not\parallel}$. This is shown in Table 3.2.

Note that the omission of tadpoles or multiple edges does not alter the Euler characteristic in connected case, except for one class in odd case. This of course follows from Proposition 2.22 ([21, Proposition 3.4]) and from Theorem 4.2 as is expected, but we nevertheless provide the computed data below as a consistency check.

Remark 3.7. Note in particular that the Euler characteristics of the even and odd graph complexes are astonishingly similar, up to a conventional sign factor.

$\overline{\dim}_{v,e}^{+-+\geq 3}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
36	0	0	0	0	0	0	0	0	0	0	323	6914443	10238197936	2434452003889	149691529204800	3065797882254941	24210784546884942	80125688022289219	115536095318709248	72644240367676642	19131543576030878	1915629738270074	59880793175801	377177693363	103765564
35	0	0	0	0	0	0	0	0	0	0	1005	12300248	11395934706	1854245794476	81248861159249	1206602153008407	6924193168254424	16471150498499870	16647312912300163	7021476589491775	1152712839432828	63299869884371	840308953765	1094185703	0
34	0	0	0	0	0	0	0	0	0	0	2853	20143069	11797416222	1312699826317	40738291624423	434093265619034	1782503996695968	2981432508808535	2045883700715935	551569517056290	52153482139678	1353903801649	5279400293	0	0
33	0	0	0	0	0	0	0	0	0	0	7858	30373223	11324845175	859478584099	18738781553491	141412445181865	407651323249567	466336702482262	208454338524126	33628719414227	1628346636468	15426982080	6247329	0	0
32	0	0	0	0	0	0	0	0	0	0	20198	42109841	10041540904	517319664519	7841523412407	41233491145291	81471527376402	61513198271050	16943477036119	1487486705470	30453218431	59572065	0	0	0
31	0	0	0	0	0	0	0	0	0	0	46221	53552162	8183597882	284157684349	2954983872822	10607822460710	13933178693051	6623831313164	1040005051656	42886529623	257189208	0	0	0	0
30	0	0	0	0	0	0	0	0	0	3	93366	62298561	6092177954	141175882137	990244991515	2364327746626	1983424893300	556534619930	44349201196	664231574	412966	0	0	0	0
29	0	0	0	0	0	0	0	0	0	32	168907	66042289	4111001284	62747495599	290471343715	445893643989	226333613711	34141128298	1141986095	3521433	0	0	0	0	0
28	0	0	0	0	0	0	0	0	1	127	276458	63413970	2490799385	24610685274	73085889211	68938018460	19620090166	1375793368	13412419	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	1	311	407915	54684111	1339061158	8371041155	15357400905	8360542063	1190269491	30068745	30528	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	684	537528	41907083	629274575	2413785991	2598217463	745263236	43995862	227577	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0	0	1545	627110	28187228	253575578	572310253	335860329	44006088	749459	0	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0	0	1	3213	641553	16379726	85417799	106923865	30618203	1424662	2526	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	7	5625	567604	8048887	23223881	14779073	1718769	16244	0	0	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0	17	8208	426113	3246800	4844186	1365037	45030	0	0	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0	28	10175	265083	1032036	716981	69847	221	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	65	10658	132526	243759	66081	1274	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	155	9033	50863	38916	2938	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	252	5849	13867	3497	30	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	291	2742	2329	110	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	4	262	879	188	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	1	14	179	170	7	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	1	16	75	13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	10	12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	6	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$\overline{\dim}_{v,e}^{-+-\geq 3}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
36	0	0	0	0	0	0	0	0	0	1	478	7184889	10264841321	2428566238061	149215526673547	3058791031411277	24190059107290419	80161353715143616	115688062454536997	72766334437488265	19162018671075010	1917876403039649	59916327855165	377215786591	104035119
35	0	0	0	0	0	0	0	0	0	0	1476	12656450	11409106391	1848695699280	80990011921765	1204350598200103	6922376949016458	16487604325642019	16675515572742650	7034270197508544	1154435830441245	63356370706979	840549193304	1094411650	0
34	0	0	0	0	0	0	0	0	0	1	3782	20544991	11795953743	1308152524469	40616202657361	433537414482164	1783236020595934	2986023303866542	2049945027230478	552578062244414	52218942157741	1354622849653	5278660177	0	0
33	0	0	0	0	0	0	0	0	0	1	9999	30816037	11308785524	856238088390	18690570825206	141336375950228	408109763954217	467278000530782	208900734518827	33684818059654	1629759318748	15428831783	6302088	0	0
32	0	0	0	0	0	0	0	0	0	5	22946	42496925	10015495299	515352276433	7826655271164	41247052313556	81620319199596	61659409155502	16979226824941	1489440289747	30467187002	59607306	0	0	0
31	0	0	0	0	0	0	0	0	0	6	50341	53888311	8153609941	283162797859	2952125602278	10621286120700	13967209256759	6640671981615	1041935068084	42921726596	257014269	0	0	0	0
30	0	0	0	0	0	0	0	0	0	26	98122	62477351	6065877636	140785143630	990407963724	2369520890328	1989161366018	557901090988	44407460914	664289471	424465	0	0	0	0
29	0	0	0	0	0	0	0	0	0	49	176937	66063551	4092424661	62644345618	290887773316	447239696071	227024876844	34209898447	1142991182	3528730	0	0	0	0	0
28	0	0	0	0	0	0	0	0	1	165	284646	63216765	2481023713	24607633969	73281944226	69183841838	19676613873	1377692570	13378933	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	345	416756	54431646	1335414915	8384065789	15415999361	8391997470	1192670365	30060258	32701	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	1	899	541142	41679653	628888950	2422021662	2609841863	747657371	44083731	228906	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0	1	1768	629051	28088250	253989803	575085565	337492002	44125758	744159	0	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0	0	5	3539	638053	16346366	85804143	107579823	30713501	1422641	3056	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	5	5655	565597	8065049	23361947	14854653	1727020	16468	0	0	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0	24	8455	423591	3261195	4888420	1373151	43891	0	0	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	1	36	10175	266543	1043984	720525	69551	364	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	100	10780	132923	245866	67271	1331	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	1	146	8828	51862	39501	2669	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	1	266	5961	13798	3406	64	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	4	273	2773	2519	126	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	4	303	955	153	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	1	14	156	154	13	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	11	84	13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	1	15	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	5	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	3	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 3.1: Dimensions of two cases of graph spaces.

b	Even				Odd			
	All		Connected		All		Connected	
	$\chi_{b,even}$	$\chi^{\not\circlearrowleft}_{b,even}$	$\overline{\chi}_{b,even}$	$\overline{\chi}^{\not\circlearrowleft}_{b,even}$	$-\chi_{b,odd}$	$-\chi^{\not\Leftrightarrow}_{b,odd}$	$-\overline{\chi}_{b,odd}$	$-\overline{\chi}^{\not\Leftrightarrow}_{b,odd}$
1	0	0	0	0	1	0	1	0
2	1	1	1	1	2	1	1	1
3	0	0	0	0	3	1	1	1
4	2	2	1	1	6	3	2	2
5	-1	-1	-1	-1	8	2	1	1
6	3	3	1	1	14	6	2	2
7	-1	-1	0	0	20	6	2	2
8	4	4	0	0	32	12	2	2
9	-4	-4	-2	-2	44	12	1	1
10	6	6	1	1	68	24	3	3
11	-5	-5	0	0	93	25	1	1
12	8	8	0	0	139	46	3	3
13	-10	-10	-2	-2	191	52	4	4
14	12	12	0	0	274	83	2	2
15	-18	-18	-4	-4	372	98	2	2
16	12	12	-3	-3	529	157	6	6
17	-25	-25	-1	-1	713	184	4	4
18	28	28	8	8	980	267	-5	-5
19	-25	-25	12	12	1300	320	-14	-14
20	62	62	27	27	1759	459	-21	-21
21	-22	-22	14	14	2318	559	-11	-11
22	56	56	-25	-25	3119	801	21	21
23	-74	-74	-39	-39	4107	988	44	44
24	-396	-396	-496	-496	5914	1807	504	504
25	-3068	-3068	-2979	-2979	10508	4594	2969	2969
26	-794	-794	-412	-412	13606	3098	413	413
27	35619	35619	38725	38725	-18948	-32554	-38717	-38717
28	9349	9349	10583	10583	-21109	-2161	-10578	-10578
29	-634587	-634587	-667610	-667610	622510	643619	667596	667596
30	39755	39755	28305	28305	560813	-61697	-28290	-28290

Table 3.2: The table of the Euler characteristics of the various graph complexes.

Chapter 4

Graphs with multiple hairs and multiple edges may be omitted

In this chapter we show that cohomology almost does not change if we disallow multiple hairs and multiple edges. It is an extension of [22, Theorem 2] where we showed the claim about multiple edges only for hairless complexes.

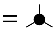
4.1 Multiple hairs

Recall that multiple hairs are possible only if $\kappa = +$, that is for $m - n$ odd.

Theorem 4.1. *For $m - n$ odd and for every constraint* $\mathrm{sup} \subset \{\geq i, \circ, \not\leftarrow\}$, $i \in \{1, 2, 3\}$ *it holds that*

$$H\left(\mathrm{HGC}_{m,n}^{\mathrm{sup}}, \delta\right) = H\left(\mathrm{HGC}_{m,n}^{\mathrm{sup}\not\leftarrow}, \delta\right) \oplus \mathbb{K}[2n - 2m - 3].$$

The class in degree $2m - 2n + 3$ is

(4.1) $\sigma_3 :=$  $\in \mathrm{HGC}_{m,n}.$

Proof. We need to prove that

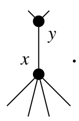
$$H\left(\mathrm{HGC}_{m,n}^{\mathrm{sup}\not\leftarrow} \oplus \mathbb{K}\sigma_3, \delta\right) = H\left(\mathrm{HGC}_{m,n}^{\mathrm{sup}}, \delta\right).$$

Since the former complex is a sub-complex of the latter one, it is enough to prove that the quotient

$$(\mathrm{C}, \delta) := \left(\mathrm{HGC}_{m,n}^{\mathrm{sup}}, \delta\right) / \left(\mathrm{HGC}_{m,n}^{\mathrm{sup}\not\leftarrow} \oplus \mathbb{K}\sigma_3, \delta\right) \tag{4.2}$$

is acyclic. The quotient (C, δ) is actually a complex similar to $\left(\mathrm{HGC}_{m,n}^{\mathrm{sup}}, \delta\right)$ but with graphs that do not have multiple hairs and σ_3 identified to zero.

Let a *flower* on vertex y with a root x be sub-graph consisting of a trivalent vertex y that has two hairs, its hairs, edge and another end x of the edge, like in the diagram:

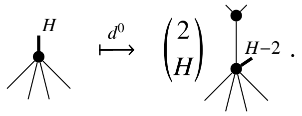


Here a short line starting at a vertex represents a hair, and longer lines starting at a vertex represent possible other edges or hairs. The graph

is considered as one flower.

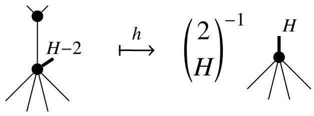
Note that the differential can not produce two flowers. We set up a spectral sequence on (C, δ) on the number of vertices minus the number of flowers. With splitting the complex according to the number of hairs, Corollary

B.3 implies that the spectral sequence converges correctly. Since the differential always creates one vertex, the first differential of this spectral sequence δ^0 is the one that creates a flower:



Here, small thick line with a letter H that starts on a vertex means H hairs. One needs to check that the above action is not canceled by a_x of e_k from the definition of the differential (2.9). But this can happen only if the whole graph is a vertex with two hairs, being excluded by the valence condition, or σ_3, being identified with zero. Note that by symmetry two flowers with the same root are not possible.

We can define the homotopy going in the opposite direction:



One now easily checks that for a graph $\Gamma \in \mathrm{C}$

$$\left(\delta^0 h + h\delta^0\right)\Gamma = c\Gamma.$$

where $c > 0$ if Γ has a multiple hairs. This implies that the fist page of the spectral sequence is acyclic, and hence the whole complex. That was to be demonstrated. □

4.2 Multiple edges

Recall that multiple edges are possible only if $\mu = +$, that is for n odd.

Theorem 4.2. *For n odd it holds that*

$$H(\mathrm{GC}_n, \delta) = H\left(\mathrm{GC}_n^{\not\Leftarrow}, \delta\right) \oplus \mathbb{K}[2n-3], \tag{4.3}$$

for n odd, m even it holds that

$$H(\mathrm{HGC}_{m,n}, \delta) = H\left(\mathrm{HGC}_{m,n}^{\not\Leftarrow}, \delta\right) \oplus \mathbb{K}[2n-4], \tag{4.4}$$

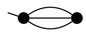
and for n and m odd it holds that

$$H(\mathrm{HGC}_{m,n}, \delta) = H\left(\mathrm{HGC}_{m,n}^{\not\Leftarrow}, \delta\right) \oplus \mathbb{K}[2n-4] \oplus \mathbb{K}[2n-m-4]. \tag{4.5}$$

The class in degree $3-2n$ is

(4.6) $\Theta :=$  $\in \mathrm{GC}_n.$

It can be considered as a graph in $\mathrm{fHGC}_{m,n}$ and there it has degree $3-n-m$. The class in degree $4-2n$ is

(4.7) $\Theta_h :=$  $\in \mathrm{HGC}_{m,n},$

and the class in degree $4+m-2n$ is

(4.8) $\Phi :=$  $\in \mathrm{HGC}_{m,n}$

what is not zero only in the case n and m are odd because of the symmetry.

Proof. Both claims of the theorem together say (cf. (2.12))

$$H\left(\mathrm{fHGCc}_{m,n}^{\geq 3}, \delta\right) = H\left(\mathrm{fHGCc}_n^{\geq 3 \not\Leftarrow}, \delta\right) \oplus \mathbb{K}[m+n-3] \oplus \mathbb{K}[2n-4] \oplus \begin{cases} 0 & \text{for } m \text{ even,} \\ \mathbb{K}[2n-m-4] & \text{for } m \text{ odd.} \end{cases}$$

We actually prove that

$$H\left(\mathrm{fHGCn}_{m,n}^{\geq 3 \not\succ}, \delta\right) = H\left(\mathrm{fHGCn}_{m,n}^{\geq 3 \not\Leftarrow \not\succ}, \delta\right) \oplus \mathbb{K}[m+n-3] \oplus \mathbb{K}[2n-4] \oplus \begin{cases} 0 & \text{for } m \text{ even,} \\ \mathbb{K}[2n-m-4] & \text{for } m \text{ odd,} \end{cases}$$

and use Proposition 2.28 and Theorem 4.1. The former complex splits as

$$\left(\mathrm{fHGCn}_{m,n}^{\geq 3 \not\curlywedge}, \delta\right) = (\mathrm{C}, \delta) \oplus (\mathbb{K}\Theta, 0) \oplus (\mathbb{K}\Theta_h, 0) \oplus \begin{cases} 0 & \text{for } m \text{ even,} \\ (\mathbb{K}\Phi, 0) & \text{for } m \text{ odd,} \end{cases}$$

where $\mathrm{C} \subset \mathrm{fHGCn}_{m,n}^{\geq 3 \not\curlywedge}$ is spanned by all graphs except Θ, Θ_h and Φ. It is clear that $H(\mathbb{K}\Theta) = \mathbb{K}[m+n-3]$, $H(\mathbb{K}\Theta_h) = \mathbb{K}[2n-4]$ and $H(\mathbb{K}\Phi) = \mathbb{K}[2n-m-4]$ for m odd, so the claim reduces to showing that

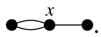
$$H(\mathrm{C}, \delta) = H\left(\mathrm{fHGCn}_{m,n}^{\geq 3 \not\Leftarrow \not\curlywedge}, \delta\right). \tag{4.9}$$

Recall that a *multiple edge* e is the pair of vertices (x, y) such that there are more than one edge connecting them. Let $N(e)$ be the number of edges in multiple edge e, let $S(e) := \left\lfloor \frac{N(e)}{2} \right\rfloor$ be its *strength* and let the *total strength* $S(\Gamma)$ be the sum of the strengths of all multiple edges in a graph Γ. The differential δ can not increase the total strength, so we have a filtration of C on the total strength. With splitting the complex according to the number of hairs, Corollary B.3 implies that the spectral sequence converges correctly.

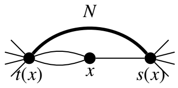
The differential δ^0 on the first page does not decrease the total strength:

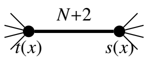
$$\delta^0(\Gamma) = \sum_{x \in V(\Gamma)} \left(\frac{1}{2} sps_x(\Gamma) - a_x(\Gamma)\right) - \sum_{k \in H(\Gamma)} e_k(\Gamma), \tag{4.10}$$

where sps_x stands for "strength preserving splitting of x" and means part of the splitting s_x that do not split even-fold multiple edge in two parts with odd number of edges.

Let a *good* vertex be a hairless trivalent vertex x two of whose edges form a double edge: 

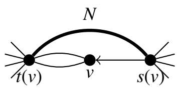
. The other end of the double edge is denoted by $t(x)$ and the other end of the single edge is denoted by $s(x)$. We require $t(x) \neq s(x)$. For every good vertex x we define a map h_x such that locally:

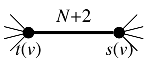
(4.11) 

$\overset{h_x}{\longmapsto} \pm \begin{cases} \frac{1}{N+2} & \text{if } N \text{ odd} \\ \frac{1}{N+1} & \text{if } N \text{ even} \end{cases}$ 

,

where the thick edge with number $N \geq 0$ indicates an N-fold edge, and the sign is chosen such that



$\overset{h_x}{\longmapsto} + \begin{cases} \frac{1}{N+2} & \text{if } N \text{ odd} \\ \frac{1}{N+1} & \text{if } N \text{ even} \end{cases}$ 

,

if v is the last vertex, all other vertices keep their number and all edges keep their orientation. Note that h_x does not change the total strength. We put $h_x = 0$ if x is not good, and

$$h\Gamma = \sum_{x \in V(\Gamma)} h_x \Gamma. \tag{4.12}$$

Lemma 4.3. *For every graph* $\Gamma \in \mathrm{C}$

$$\left(\delta^0 h + h\delta^0\right)\Gamma = 2S(\Gamma)\Gamma.$$

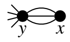
Proof. For $y \in V(\Gamma)$ let

$$\delta_y^0(\Gamma) := \frac{1}{2} sps_x(\Gamma) - a_x(\Gamma) - \sum_{\substack{k \in H(\Gamma) \\ \Gamma_H(k) = y}} e_k(\Gamma), \tag{4.13}$$

such that $\delta^0(\Gamma) = \sum_{y \in V(\Gamma)} \delta_y^0(\Gamma)$. We compute

$$\left(\delta^0 h + h\delta^0\right)\Gamma = \sum_{y \in V(h(\Gamma))} \delta_y^0 \sum_{x \in V(\Gamma)} h_x \Gamma + \sum_{x \in V(\delta^0(\Gamma))} h_x \sum_{y \in V(\Gamma)} \delta_y^0 \Gamma = \sum_{\substack{x, y \in V(\Gamma) \\ x \neq y}} \left(\delta_y^0 h_x \Gamma + h_x \delta_y^0 \Gamma\right) + 2 \sum_{x \in V(\Gamma)} h_x \delta_x^0 \Gamma. \tag{4.14}$$

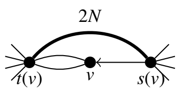
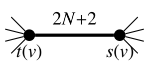
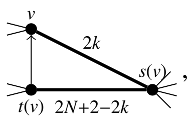
We claim that δ_y^0 can not change the property of being good of a vertex $x \neq y$. Clearly, δ_y^0 can not change a good vertex x to become not good. On the other hand, it can not affect non-neighbors, can not change the valence of other vertices, can not manipulate hairs on other vertices, and can not produce multiple edge. Therefore, if δ_y^0

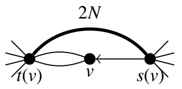
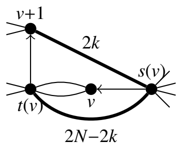
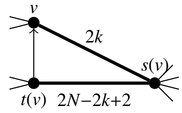
makes x good, x was already hairless and trivalent with two of the edges pointing to the same vertex before acting of δ^0_y. The only possibility when x was not good before acting is when x was a trivalent neighbour of y all of whose three edges form a triple edge towards y, that is  . But y can not be a separating vertex and hence can not be connected to anything else than x, with possibly hairs. Multiple hairs are not allowed, with one hair the whole graph is Θ_h and with no hairs the whole graph is Θ. But both those graphs have been explicitly excluded. Therefore

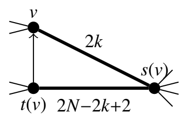
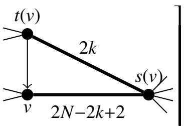
$$(4.15)\quad \left(\delta^0 h + h\delta^0\right)\Gamma = \sum_{\substack{x\in V(\Gamma)\\ x \text{ good}}}\ \sum_{\substack{y\in V(\Gamma)\\ y\neq x}} \left(\delta^0_y h_x\Gamma + h_x\delta^0_y\Gamma\right) + 2\sum_{x\in V(\Gamma)} h_x\delta^0_x\Gamma =$$

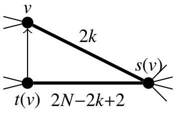
$$= \sum_{\substack{x\in V(\Gamma)\\ x \text{ good}}}\ \sum_{\substack{y\in V(\Gamma)\\ y\notin\{x,t(x),s(x)\}}} \left(\delta^0_y h_x\Gamma + h_x\delta^0_y\Gamma\right) + \sum_{\substack{x\in V(\Gamma)\\ x \text{ good}}} \left(\delta^0_{t(x)} h_x\Gamma + h_x\delta^0_{t(x)}\Gamma\right) + \sum_{\substack{x\in V(\Gamma)\\ x \text{ good}}} \left(\delta^0_{s(x)} h_x\Gamma + h_x\delta^0_{s(x)}\Gamma\right) + 2\sum_{x\in V(\Gamma)} h_x\delta^0_x\Gamma.$$

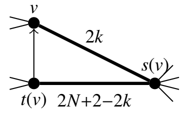
The first term is trivially zero. We claim that the second term is also zero. It is enough to assume that x is the last vertex v. We consider separately the cases of odd and even numbers of "bridging" edges between $s(v)$ and $t(v)$. First, in the even bridging:

 $\overset{h_v}{\longmapsto}\ \dfrac{1}{2N+1}$  $\overset{\delta^0_{t(v)}}{\longmapsto}\ \dfrac{1}{4N+2}\displaystyle\sum_{k=0}^{N+1}\binom{2N+2}{2k}\sum_{conn}$  ,

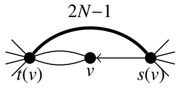
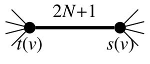
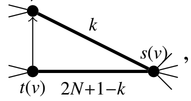
 $\overset{\delta^0_{t(v)}}{\longmapsto}\ \displaystyle\sum_{k=0}^{N}\binom{2N}{2k}\sum_{conn}$  $\overset{h_v}{\longmapsto}\ -\displaystyle\sum_{k=0}^{N}\binom{2N}{2k}\frac{1}{2N-2k+1}\sum_{conn}$  $=$

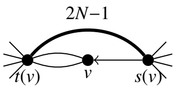
$= -\dfrac{1}{2}\Bigg[\displaystyle\sum_{k=0}^{N+1}\binom{2N}{2k}\frac{1}{2N-2k+1}\sum_{conn}$  $+\displaystyle\sum_{k=0}^{N+1}\binom{2N}{2k}\frac{1}{2N-2k+1}\sum_{conn}$  $\Bigg] =$

$= -\dfrac{1}{2}\displaystyle\sum_{k=0}^{N+1}\left[\binom{2N}{2k}\frac{1}{2N-2k+1} + \binom{2N}{2N-2k+2}\frac{1}{2k-1}\right]\sum_{conn}$  $=$

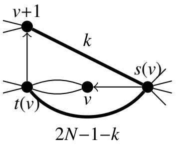
$= \dfrac{-1}{4N+2}\displaystyle\sum_{k=0}^{N+1}\binom{2N+2}{2k}\sum_{conn}$  ,

where $\sum_{conn}$ is the sum over all possibilities of connecting remaining edges of $v(t)$ to new vertices. We have omitted the terms $a_{t(v)}$ and e_k for hairs on $t(v)$ in the action of $\delta^0_{t(v)}$, but they trivially cancel. For an odd number of bridging edges the situation is similar:

 $\overset{h_v}{\longmapsto}\ \dfrac{1}{2N+1}$  $\overset{\delta^0_{t(v)}}{\longmapsto}\ \dfrac{1}{4N+2}\displaystyle\sum_{k=0}^{2N+1}\binom{2N+1}{k}\sum_{conn}$  ,

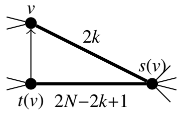


$$\overset{\delta^0_{t(v)}}{\longmapsto} \sum_{k=0}^{2N-1}\binom{2N-1}{k}\sum_{conn}$$

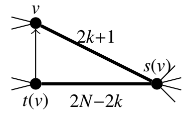


$$\overset{h_v}{\longmapsto}$$

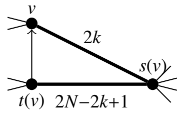
$$\overset{h_v}{\longmapsto} \quad -\sum_{k=0}^{N-1}\binom{2N-1}{2k}\frac{1}{2N-2k+1}\sum_{conn}$$



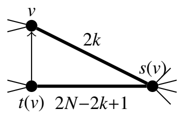
$$-\sum_{k=0}^{N-1}\binom{2N-1}{2k+1}\frac{1}{2N-2k-1}\sum_{conn}$$



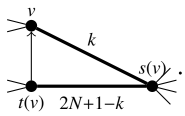
$$= -\sum_{k=0}^{N}\left[\binom{2N-1}{2k}\frac{1}{2N-2k+1}+\binom{2N-1}{2N-2k+1}\frac{1}{2k-1}\right]\sum_{conn}$$



$$= \frac{-1}{2N+1}\sum_{k=0}^{N}\binom{2N+1}{2k}\sum_{conn}$$



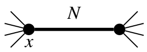
$$= \frac{-1}{4N+2}\sum_{k=0}^{2N+1}\binom{2N+1}{k}\sum_{conn}$$



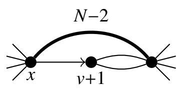
Interchanging with , $s(v)$ with $t(v)$ and vice versa in the above calculations leads to the conclusion that the third term in (4.15) is $\delta^0_x h_x\Gamma + h_x\delta^0_{s(x)}\Gamma = 0$. The remaining term is:

$$\left(\delta^0 h + h\delta^0\right)\Gamma = 2\sum_{x\in V(\Gamma)} h_x\delta^0_x\Gamma = 2\sum_{x\in V(\Gamma)} h_{v+1}\delta^0_x\Gamma. \tag{4.16}$$

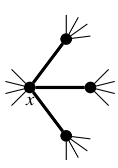
It suffices to consider terms for which δ^0_x makes the new vertex $v+1$ good, otherwise the term is zero. Vertex $v+1$ in $\delta^0_x\Gamma$ has a single edge towards x, so it is good if and only if a multiple edge e has been split into a double edge heading towards $v+1$ and an $(N(e)-2)$-fold edge heading towards the new x, and all other edges heading towards the new x, i.e.



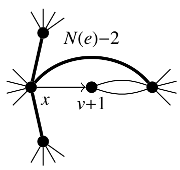
$$\longmapsto \quad \frac{1}{2}\binom{N}{2}$$



We have to check that this term is not canceled by a_x or e_k for a hair k on x, i.e. that it is not at the same time true that $N = 2$ and that x does not have other edges. But in that case, since the other end of the N-fold edge is not separating, the whole graph would be $= 0$ or or . The second graph is not in C since a vertex is not at least 3-valent, and the third graph is zero for m even and explicitly excluded from C for m odd. Therefore

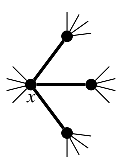


$$\overset{\delta^0_x}{\longmapsto} \sum_{e \text{ multiple edge at } x}\frac{1}{2}\binom{N(e)}{2}$$

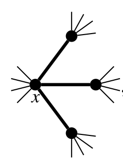


$$+ \text{(something where } v+1 \text{ is not good)} \overset{h_{v+1}}{\longmapsto}$$

$$\overset{h_{v+1}}{\longmapsto} \sum_{e \text{ multiple edge at } x}\frac{1}{2}\binom{N(e)}{2}\begin{cases}\frac{1}{N(e)} & \text{if } N(e) \text{ odd}\\ \frac{1}{N(e)-1} & \text{if } N(e) \text{ even}\end{cases}$$



$$= \frac{1}{2}\sum_{e \text{ multiple edge at } x} S(e)$$



,

$$\left(\delta^0 h + h\delta^0\right)\Gamma = 2\sum_{x\in V(\Gamma)} h_{v+1}\delta^0_x\Gamma = 2\sum_{x\in V(\Gamma)}\frac{1}{2}\sum_{e \text{ multiple edge at } x} S(e)\Gamma = 2\sum_{e \text{ multiple edge}} S(e)\Gamma = 2S(\Gamma)\Gamma. \tag{4.17}$$

□

The lemma ensures that all rows on the first page of our spectral sequences are exact, unless the total strength is 0, that is we are in the last subcomplex $\mathrm{fHGCn}_{m,n}^{\geq 3 \not\leftrightarrow \not\rightarrow}$ of the filtration. Therefore, on the second page, there are all zeros except in the last row where there is $H(\mathrm{fHGCn}_{m,n}^{\geq 3 \not\leftrightarrow \not\rightarrow})$, so the spectral sequences converges to $H(\mathrm{fHGCn}_{m,n}^{\geq 3 \not\leftrightarrow \not\rightarrow})$. That was to be demonstrated. □

Chapter 5

Extra differential for graph complex, n even

In this chapter we study the cohomology of $\left(\mathrm{fGCc}_n^{\circlearrowleft}, \delta\right)$ for even n. As a representative of even n we work with the case $n = 0$. In this case the degree of a graph is simply the number of edges. We introduce an another differential on $\mathrm{fGCc}_0^{\circlearrowleft}$ that fits well with the standard differential, leading to an improvement in understanding the standard cohomology. The result is from [7], with an improvement from [23].

5.1 Adding an edge

Definition 5.1. Let $\nabla : \mathrm{fGC}_0 \to \mathrm{fGC}_0$ be defined as the Lie bracket with the "tadpole" graph:

$$\nabla := [\ \circlearrowleft\ , \cdot\ ].$$

An equivalent definition on a graph Γ is

$$\nabla(\Gamma) := \sum_{\substack{x,y \in V(\Gamma) \\ x \neq y}} \nabla_{x,y}(\Gamma),$$

where $\nabla_{x,y}$ adds an edge between x and y.

Remark 5.2. One easily checks that ∇ is well defined on all sub-complexes $\mathrm{fGC}_0^{\mathrm{sup}}$ and also for connected sub-complexes $\mathrm{fGCc}_0^{\mathrm{sup}}$ for any constraint $\mathrm{sup} \subset \{\geq i, \circlearrowleft, \not\Leftarrow, \not\succ\}$, $i \in \{1, 2, 3\}$.

Proposition 5.3. *On* fGC_0 *it holds that*

- $\nabla^2 = 0,$
- $\delta\nabla + \nabla\delta = 0,$
- $(\delta + \nabla)^2 = 0.$

Proof. Easy verification. □

Since the degree of ∇ is 1, the proposition implies that ∇ and $\delta + \nabla$ are both differentials on $\mathrm{fGC}_n^{\circlearrowleft}$.

Proposition 5.4. $H\left(\mathrm{fGC}_0^{\circlearrowleft}, \nabla\right)$ *is one-dimensional, the class being represented by the single vertex graph* σ.

Proof. We have by definition

$$\mathrm{fGC}_0^{\circlearrowleft} = \prod_{v \geq 1} \mathrm{V}_v \mathrm{fGC}_0^{\circlearrowleft} = \prod_{v \geq 1} \left(\bar{\mathrm{V}}_v \mathrm{fGC}_0^{\circlearrowleft}\right)^{S_v}.$$

The differential ∇ does not change the number of vertices, so Proposition A.1 implies that

$$H(\mathrm{fGC}_0^{\circlearrowleft}, \nabla) = \prod_{v \geq 1} \left(H\left(\bar{\mathrm{V}}_v \mathrm{fGC}_0^{\circlearrowleft}, \nabla\right)\right)^{S_v}.$$

Therefore, it suffices to calculate $H\left(\bar{\mathrm{V}}_v\mathrm{fGC}_0^{\circlearrowleft},\nabla\right)$. For $v = 1$ it is clearly 1-dimensional. For $v \geq 2$ there is an explicit homotopy $h : \bar{\mathrm{V}}_v\mathrm{fGC}_0^{\circlearrowleft} \to \bar{\mathrm{V}}_v\mathrm{fGC}_0^{\circlearrowleft}$:

$$h\Gamma = \begin{cases} \pm(\Gamma - (1,2)) & \text{if there is an edge between vertices 1 and 2} \\ 0 & \text{otherwise,} \end{cases}$$

where $\Gamma - (1,2)$ is the graph Γ without the edge between vertices 1 and 2 and the sign is chosen + if $(1,2)$ is the first edge in the ordering before the action of S_e. We find that $\nabla \circ h + h \circ \nabla = id$, and hence $H\left(\bar{\mathrm{V}}_v\mathrm{fGC}_0^{\circlearrowleft},\nabla\right) = 0$.

We also remark that an alternative, "more symmetric" proof may be obtained by defining a homotopy $\tilde{h}$, such that

$$\tilde{h}\Gamma = \sum_e \pm(\Gamma - \{e\})$$

is the sum over all ways of removing one edge from the graph Γ. Then $(\nabla\tilde{h} + \tilde{h}\nabla)\Gamma = \frac{v(v-1)}{2}\Gamma$, and hence we arrive at the same conclusion as before. □

Corollary 5.5. *$H(\mathrm{fGCc}_0^{\circlearrowleft},\nabla)$ is two-dimensional, the classes being represented by the single vertex graph σ and the graph $\lambda = \bullet\!\!-\!\!\bullet$.*

Proof. Again, it suffices to consider the sub-complexes for fixed number of vertices $\mathrm{V}_v\mathrm{fGCc}_0^{\circlearrowleft}$. Clearly $\mathrm{V}_1\mathrm{fGCc}_0^{\circlearrowleft}$ and $\mathrm{V}_2\mathrm{fGCc}_0^{\circlearrowleft}$ are 1 dimensional and hence the differential acts trivially.

We assume inductively that for all $3 \leq j \leq v-1$ we have $H\left(\mathrm{V}_v\mathrm{fGCc}_0^{\circlearrowleft},\nabla\right) = 0$. Let Γ be a cocycle in $\mathrm{V}_v\mathrm{fGC}_0^{\circlearrowleft}$, i.e. $\nabla\Gamma = 0$. Let $\nu \in \mathrm{V}_v\mathrm{fGC}_0^{\circlearrowleft}$, such that $\nabla\nu = \Gamma$. We decompose ν into

$$\nu = \mathrm{C}_1\nu + \mathrm{C}_2\nu + \cdots + \mathrm{C}_k\nu.$$

where $\mathrm{C}_l\nu$ is the term of graphs with l connected components. We chose ν such that k is least possible. If $k = 1$, ν is connected and Γ is a exact in $\mathrm{V}_v\mathrm{fGC}_0^{\circlearrowleft}$ what we need. So let us assume $k > 1$.

We split $\nabla = \nabla_0 + \nabla_1$ on $\mathrm{V}_v\mathrm{fGC}_0^{\circlearrowleft}$ into a term that leaves the number of connected components invariant, and one that lowers the number of connected components (necessarily by one). The leading term $\mathrm{C}_k\nu$ must be ∇_0-closed. It holds that

$$\left(\mathrm{V}_v\mathrm{fGC}_0^{\circlearrowleft},\nabla_0\right) = S^+\left(\mathrm{V}_v\mathrm{fGCc}_0^{\circlearrowleft},\nabla\right),$$

so every connected component in $\mathrm{C}_k\nu$ is ∇-closed. If one of them is exact we may replace it with its pre-image and get $\mathrm{C}_k\nu'$. Then $\mathrm{C}_k\nu = \nabla_0\mathrm{C}_k\nu'$ is a ∇_0-coboundary and we may replace ν with the cohomologous cycle $\nu - \nabla\mathrm{C}_k\nu'$ that does not have a $\mathrm{C}_k\nu$ part, contradicting the minimality of k.

So every connected component represents a non-trivial cohomology class, i.e., by the induction hypothesis, every connected component is an isolated vertex or an isolated edge between two vertices. Therefore, and because of symmetry reasons, $\mathrm{C}_k\nu$ is a linear combination of graphs which are a union of isolated vertices, and possibly at most one isolated edge.

If there is an isolated edge in $\mathrm{C}_k\nu$, for ν' being the union of all isolated vertices, ν can be replaced with $\nu - \nabla\nu'$, that again does not have a $\mathrm{C}_k\nu$ part contradicting the minimality of k. So there may only be v isolated vertices and $k = v$. But the equality $\nabla_1\mathrm{C}_v\nu + \nabla_0\mathrm{C}_{v-1}\nu = 0$ can be satisfied only if $\mathrm{C}_v\nu = 0$. Hence again $\mathrm{C}_k\nu = 0$, concluding the proof. □

Corollary 5.6. *$H\left(\mathrm{fGC}_0^{\geq 1\circlearrowleft},\nabla\right)$ is 1-dimensional, the class being represented by λ.*

Proof. Corollary 5.5 clearly implies that $H\left(\mathrm{fGCc}_0^{\geq 1\circlearrowleft},\nabla\right)$ is one-dimensional, the class being represented by λ. On $\mathrm{fGC}_0^{\geq 1}$ we set up a spectral sequence on the number of connected components. The first differential does not change that number, so the cohomology is the symmetric product of λ. But there can not be more than one λ because of the symmetry reasons, so the only class remaining is connected graph λ, concluding the proof. □

Corollary 5.7. *It holds that*

$$H\left(\mathrm{fGCc}_0^{\circlearrowleft},\delta + \nabla\right) = 0.$$

Proof. On $H\left(\mathrm{fGCc}_0^{\not\circlearrowleft}, \delta + \nabla\right)$ we set up a spectral sequence on the number of vertices. Proposition B.1 implies that the spectral sequence converges correctly.

On the first page we have $H\left(\mathrm{fGCc}_0^{\not\circlearrowleft}, \nabla\right)$, which is generated by two generators, the single vertex graph σ and the graph with two connected vertices λ (Corollary 5.5).

The differential on this page is induced by δ, mapping σ to λ. Hence the next page in the spectral sequence is zero. That finishes the proof. □

Corollary 5.8.

1. *$H\left(\mathrm{fGCc}_0^{\geq 2\not\circlearrowleft}, \delta + \nabla\right) = 0$;*

2. *$H\left(\mathrm{fGCc}_0^{\geq 1\not\circlearrowleft}, \delta + \nabla\right)$ is one-dimensional, the class being represented by $\lambda = \bullet\!\!-\!\!\!-\!\!\bullet$;*

3. *$H\left(\mathrm{fGC}_0^{\geq 1\not\circlearrowleft}, \delta + \nabla\right)$ is one-dimensional, the class being represented by λ.*

Proof.

1. On the mapping cone of the inclusion $\left(\mathrm{fGCc}_0^{\geq 2\not\circlearrowleft}, \delta + \nabla\right) \hookrightarrow \left(\mathrm{fGCc}_0^{\not\circlearrowleft}, \delta + \nabla\right)$ we set up a spectral sequence on the number $b = e - v$. The complex with the first differential is the mapping cone of the inclusion $\left(\mathrm{fGCc}_0^{\geq 2\not\circlearrowleft}, \delta\right) \hookrightarrow \left(\mathrm{fGCc}_0^{\not\circlearrowleft}, \delta\right)$. It is acyclic by Proposition 2.23. The spectral sequence converges correctly because of Proposition B.2, so the whole mapping cone is acyclic. That leads to $H\left(\mathrm{fGCc}_0^{\geq 2\not\circlearrowleft}, \delta + \nabla\right) = H\left(\mathrm{fGCc}_0^{\not\circlearrowleft}, \delta + \nabla\right) = 0$.

2. We do the same for the inclusion $\left(\mathrm{fGCc}_0^{\geq 1\not\circlearrowleft}, \delta + \nabla\right) \hookrightarrow \left(\mathrm{fGCc}_0^{\not\circlearrowleft}, \delta + \nabla\right)$, and the first differential is $\left(\mathrm{fGCc}_0^{\geq 1\not\circlearrowleft}, \delta\right) \hookrightarrow \left(\mathrm{fGCc}_0^{\not\circlearrowleft}, \delta\right)$. Clearly $\mathrm{fGCc}_0^{\not\circlearrowleft} = \mathrm{fGCc}_0^{\geq 1\not\circlearrowleft} + \mathbb{K}\sigma$, where $\sigma = \bullet$ is the graph with one vertex and no edges, so cohomology of the mapping cone $\left(\mathrm{fGCc}_0^{\geq 1\not\circlearrowleft}, \delta\right) \hookrightarrow \left(\mathrm{fGCc}_0^{\not\circlearrowleft}, \delta\right)$ is one-dimensional, so is the whole mapping cone. Since $H\left(\mathrm{fGCc}_0^{\not\circlearrowleft}, \delta + \nabla\right) = 0$ it implies $H\left(\mathrm{fGCc}_0^{\geq 2\not\circlearrowleft}, \delta + \nabla\right)$ is one-dimensional. It is easily checked that the class is represented by λ.

3. On $H\left(\mathrm{fGC}_0^{\geq 1\not\circlearrowleft}, \delta + \nabla\right)$ we set up a spectral sequence on the number of connected components. It clearly converges to the cohomology of the whole complex. In the c-th row there is a complex

$$\left(\mathrm{C}_c\mathrm{fGC}_0^{\geq 1\not\circlearrowleft}, \delta + \nabla_0\right) = \left(\left(\mathrm{fGCc}_0^{\geq 1\not\circlearrowleft}, \delta + \nabla\right)^{\otimes c}\right)^{S_c}[1-c]$$

whose cohomology is

$$H\left(\mathrm{C}_c\mathrm{fGC}_0^{\geq 1\not\circlearrowleft}, \delta + \nabla_0\right) = \left(H\left(\mathrm{fGCc}_0^{\geq 1\not\circlearrowleft}, \delta + \nabla\right)^{\otimes c}\right)^{S_c}[1-c] = \left((\mathbb{K}\lambda)^{\otimes c}\right)^{S_c}[1-c]$$

Because of the symmetry reasons, there can be only one λ, so only $\left(\mathrm{C}_1\mathrm{fGC}_0^{\geq 1\not\circlearrowleft}, \delta + \nabla_0\right) = \left(\mathrm{fGCc}_0^{\geq 1\not\circlearrowleft}, \delta + \nabla\right)$ is not acyclic, and the whole complex is one dimensional. Since $(\delta + \nabla)(\lambda) = 0$, it represents the class.

□

5.2 Deleting a vertex

In this section we introduce a new operation D on the non-hairy graph complexes fGC_n which we call "deleting a vertex".

Definition 5.9. Let $D : \mathrm{fGC}_0^{\not\circlearrowleft} \to \mathrm{fGC}_0^{\not\circlearrowleft}$ be defined on graph Γ as

$$D(\Gamma) := \sum_{x \in V(\Gamma)} D_x = \sum_{x \in V(\Gamma)} (-1)^{v(x)} \tilde{D}_x$$

where $v(x)$ is the valence of the vertex x, $\tilde{D}_x$ deletes the vertex x and sums over all ways of reconnecting edges that were connected to x to the other vertices, skipping graphs with a tadpole, and $D_x = (-1)^{v(x)}\tilde{D}_x$.

We can restrict D to $\mathrm{fGC}_0^{\geq 1\not\circlearrowleft}$ and $\mathrm{fGC}_0^{\geq 2\not\circlearrowleft}$. The following propositions about D are stated in the broadest space possible, i.e. in $\mathrm{fGC}_0^{\geq i\not\circlearrowleft}$ for the smallest i where it holds.

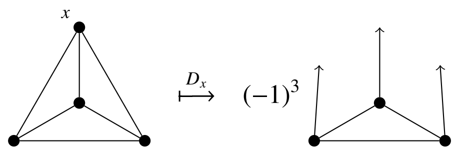


Figure 5.1: Example of the action D_x, deleting the vertex x. The graph at the right means the sum over all graphs that can be formed by attaching ends of arrows to vertices, without making a tadpole.

Proposition 5.10. *In* $\mathrm{fGC}_0^{\geq 2\,\not\circlearrowleft}$ *it holds that*

$$D^2 = 0.$$

Proof. For a graph Γ

$$D^2(\Gamma) = \sum_{\substack{x,y\in V(\Gamma)\\ x\neq y}} D_y D_x(\Gamma).$$

We now fix $x, y \in V(\Gamma)$, $x \neq y$. Since D does not change the number of edges, we can distinguish them. Vertices of $\Gamma \in \bar{\mathrm{V}}_v\mathrm{fGC}_0^{\geq 2\,\not\circlearrowleft}$ are at least 2-valent, so there exist an edge between x and another vertex that is not y. We choose one of them and call it f.

D_yD_x first deletes vertex x, reconnects its edges to other vertices, deletes vertex y and reconnects its edges, including those that came from x, to other vertices. Let us fix one way of reconnecting edges that are not f and the final destination of f. For that way of reconnecting, in D_yD_x there are two terms: the one where f goes directly to its final destination with D_x and the one where f goes to y with D_x and to its final destination with D_y. In the later the valence of y while applying D_y is by 1 bigger than its valence while applying D_y in the former term, making the terms cancel each other because of the sign dependence on valence. It follows that $D_yD_x(\Gamma) = 0$ and therefore $D^2(\Gamma) = 0$. □

Proposition 5.11. *In* $\mathrm{fGC}_0^{\geq 1\,\not\circlearrowleft}$ *it holds that*

$$D^4 = 0.$$

Proof. From the previous proof we see that $D_yD_x(\Gamma)$ may only be non-zero if x is 1-valent vertex with an edge connecting to y.

So $D_zD_yD_x(\Gamma) \neq 0$ implies that x is of that kind. But also $D_zD_y(D_x(\Gamma)) \neq 0$ implies that y is 1-valent and connected to z in $D_x(\Gamma)$. Since D_x did not change the valence of y, already in Γ it was 1-valent, and vertices x and y formed $\lambda = \bullet\!\!-\!\!\bullet$.

Now let us pick a term in $D^4(\Gamma)$, say $D_wD_zD_yD_x(\Gamma)$. It being non-zero implies $D_zD_yD_x(\Gamma) \neq 0$, so x and y form λ. Similarly, $D_wD_zD_y(D_x(\Gamma)) \neq 0$ imply y and z form λ in $D_x(\Gamma)$. Since the edge at z came from x, z was isolated in Γ what is not possible in $\mathrm{fGC}_0^{\geq 1}$. Therefore $D_wD_zD_yD_x(\Gamma)$ is always 0 and $D^4 = 0$. □

Proposition 5.12. *In* $\mathrm{fGC}_0^{\not\circlearrowleft}$ *it holds that*

$$\delta D - D\delta = \nabla.$$

Proof.

$$(\delta D(\Gamma) - D\delta(\Gamma)) = \delta\left(\sum_x D_x(\Gamma)\right) - D\left(\sum_y \frac{1}{2}s_y(\Gamma) - a_y(\Gamma)\right) = \sum_x \sum_{\substack{y\\ y\neq x}} \left(\frac{1}{2}s_yD_x(\Gamma) - a_yD_x(\Gamma)\right)$$

$$-\frac{1}{2}\sum_y\left(\sum_{\substack{x\\ x\neq y}} D_xs_y(\Gamma) + D_ys_y(\Gamma) + D_zs_y(\Gamma)\right) + \sum_y\left(\sum_{\substack{x\\ x\neq y}} D_xa_y(\Gamma) + D_ya_y(\Gamma) + D_za_y(\Gamma)\right) =$$

$$= \frac{1}{2}\sum_{\substack{x,y\\ x\neq y}} s_yD_x(\Gamma) - \sum_{\substack{x,y\\ x\neq y}} a_yD_x(\Gamma) - \frac{1}{2}\sum_{\substack{x,y\\ x\neq y}} D_xs_y(\Gamma) - \sum_x D_zs_x(\Gamma) + \sum_{\substack{x,y\\ x\neq y}} D_xa_y(\Gamma) + \sum_x D_xa_x(\Gamma) + \sum_x D_za_x(\Gamma),$$

where x and y run through $V(\Gamma)$ and z is the name of the newly added vertex.

For different $x, y \in V(\Gamma)$ it easily follows that

$$s_y D_x(\Gamma) = D_x s_y(\Gamma).$$

The term $D_z s_x(\Gamma)$ first splits a new vertex z from the vertex x, connects them with an edge, say g, and reconnects some of the edges from x to z. Afterwards it deletes vertex z and reconnects edges from it to other vertices, possibly back to x, and reconnects g also to other vertex. The final result is that some of the edges are reconnected from x to other vertices, and there is a new edge g connecting x and some other vertex. The edge g can be seen as added at the end, and before that, since number of edges is not changed, we can distinguish edges. Suppose there is an edge that at the end stays at x, and call it f. We fix one way of reconnecting all other edges. For that way of reconnecting there are two terms in $D_z s_x(\Gamma)$: one where f stays at x and one where it is reconnected to z and back to x, and they cancel each other because valence of z differ by one in them. Therefore everything what survives from $D_z s_x(\Gamma)$ is reconnecting all edges from x to other vertices and adding an edge g from x to some other vertex, say y. That is exactly the same as deleting vertex x and adding an edge at y, with the opposite sign because the valence of the vertex being deleted differs by one. So we get

$$D_z s_x(\Gamma) = -\sum_{\substack{y \\ y \neq x}} a_y D_x(\Gamma).$$

An easy argument, that is left to the reader, implies

$$D_x a_x(\Gamma) = -\sum_{\substack{y \\ y \neq x}} D_x a_y(\Gamma).$$

The last term $\sum_x D_z a_x(\Gamma)$ clearly adds an edge from x to some other vertex, so it holds that

$$\sum_x D_z a_x(\Gamma) = \nabla(\Gamma).$$

By equations obtained, all except the last term of the above expression cancel, and the claimed formula follows. □

Propositions 5.10 and 5.12 easily imply $D\nabla + \nabla D = 0$ in $\mathrm{fGC}_0^{\geq 2 \not\circ}$. But we need a bit stronger result.

Proposition 5.13. *In* $\mathrm{fGC}_0^{\geq 1 \not\circ}$ *it holds that*

$$D\nabla + \nabla D = 0.$$

Proof. $D\nabla$ puts an edge in all possible ways and then deletes a vertex, say x. If the new edge has been connected to x, it is moved to another vertex. Let finally the new edge connect vertices y and z. To that position it can come in three different ways:

- directly been connected to y and z by ∇, what is exactly the corresponding term from ∇D;
- been connected from y to x by ∇ and then moved to z by D_x, what is the negative of the term in ∇D because of the sign changes in deleting vertex x with one more valence;
- and the same from z to x, what is also the negative of the term in ∇D.

All terms in ∇D are come like this, so indeed $D\nabla = \nabla D - 2\nabla D = -\nabla D$. □

Proposition 5.14. *In* $\mathrm{fGC}_0^{\geq 1 \not\circ}$ *it holds that*

$$\nabla D^2 = 0.$$

Proof. From the proof of Proposition 5.10 the term $D_y D_x(\Gamma)$ of $D^2(\Gamma)$ may only be non-zero if x is a 1-valent vertex with an edge connecting to y. Then $D_y D_x$ deletes x, y and the edge between them, reconnects all other edges from y elsewhere and adds an edge in all possible ways. Then ∇ adds another edge in all possible ways. Adding one edge on one place and another edge on another place cancels with adding edges in the opposite order, hence the result. □

The map D can connect two disconnected components of the graph, but it can also disconnect a connected component. The latter possibility could create problems for the proofs later. Therefore we define another map as in the definition.

Definition 5.15. Let $\bar{D} : \mathrm{fGC}_0^{\not\circlearrowleft} \to \mathrm{fGC}_0^{\not\circlearrowleft}$ be defined on graph $\Gamma \in \mathrm{C}_c\mathrm{fGC}_0^{\not\circlearrowleft}$ as the projection of $D(\Gamma)$ to the subspace spanned by graphs with at most c connected components, i.e. graphs in the linear combination of $D(\Gamma)$ that have more than c connected components are skipped.

Corollary 5.16.

1. $\bar{D}^2 = 0$ *in* $\mathrm{fGC}_0^{\geq 2\,\not\circlearrowleft}$,
2. $\bar{D}^4 = 0$ *in* $\mathrm{fGC}_0^{\geq 1\,\not\circlearrowleft}$,
3. $\delta\bar{D} - \bar{D}\delta = \nabla$ *in* $\mathrm{fGC}_0^{\not\circlearrowleft}$,
4. $\bar{D}\nabla + \nabla\bar{D} = 0$ *in* $\mathrm{fGC}_0^{\geq 1\,\not\circlearrowleft}$,
5. $\nabla\bar{D}^2 = 0$ *in* $\mathrm{fGC}_0^{\geq 1\,\not\circlearrowleft}$.

$\bar{D}$ can be restricted to the connected parts $\mathrm{fGCc}_0^{\not\circlearrowleft}$, $\mathrm{fGCc}_0^{\geq 1\,\not\circlearrowleft}$ and $\mathrm{fGCc}_0^{\geq 2\,\not\circlearrowleft}$. Corollary 5.16 holds for that restrictions too.

5.3 A picture of the even graph cohomology

Definition 5.17. On $\mathrm{fGC}_0^{\geq 1\,\not\circlearrowleft}$ we define a conjugated differential:

$$\tilde{\delta} := e^{\bar{D}}(\delta + \nabla)e^{-\bar{D}}.$$

Since $\bar{D}$ is nilpotent, it is well defined. Using results of Corollary 5.16 we can calculate $\tilde{\delta}$ explicitly:

$$\tilde{\delta} = \left(\mathrm{Id} + \bar{D} + \frac{\bar{D}^2}{2} + \frac{\bar{D}^3}{6}\right)(\delta + \nabla)\left(\mathrm{Id} - \bar{D} + \frac{\bar{D}^2}{2} - \frac{\bar{D}^3}{6}\right) = \tag{5.1}$$

$$= \delta + \nabla + \bar{D}\delta + \bar{D}\nabla - \delta\bar{D} - \nabla\bar{D} - \bar{D}\delta\bar{D} + \frac{\bar{D}^2\delta}{2} - \frac{\bar{D}^2\delta\bar{D}}{2} + \frac{\delta\bar{D}^2}{2} + \frac{\bar{D}\delta\bar{D}^2}{2} = \delta + \bar{D}\nabla.$$

Corollary 5.18. *There is a spectral sequence converging to*

$$H\left(\mathrm{fGCc}_0^{\geq 2\,\not\circlearrowleft}, \tilde{\delta}\right) = 0$$

whose first page is

$$H\left(\mathrm{fGCc}_0^{\geq 2\,\not\circlearrowleft}, \delta\right).$$

Furthermore, in all spectral sequences differentials on odd pages are 0.

Proof. The first part of Corollary 5.8 says that $H\left(\mathrm{fGCc}_0^{\geq 2\,\not\circlearrowleft}, \delta + \nabla\right) = 0$. Conjugation does not change the acyclicity, so $H\left(\mathrm{fGCc}_0^{\geq 2\,\not\circlearrowleft}, \tilde{\delta}\right) = 0$.

$\bar{D}\nabla$ changes $b = e - v$ by 2, so $\tilde{\delta}$ can not change b by an odd amount. Therefore complexes with that differential can be split into the direct sum of two complexes, one with even and one with odd b:

$$\left(\mathrm{fGCc}_0^{\geq 2\,\not\circlearrowleft}, \tilde{\delta}\right) = \left(\mathrm{B}_{odd}\mathrm{fGCc}_0^{\geq 2\,\not\circlearrowleft}, \tilde{\delta}\right) \oplus \left(\mathrm{B}_{even}\mathrm{fGCc}_0^{\geq 2\,\not\circlearrowleft}, \tilde{\delta}\right).$$

On each of them we set up a spectral sequence on b. It clearly converges correctly. The first page is $H\left(\mathrm{B}_{odd}\mathrm{fGCc}_0^{\geq 2\,\not\circlearrowleft}, \delta\right)$, respectively $H\left(\mathrm{B}_{even}\mathrm{fGCc}_0^{\geq 2\,\not\circlearrowleft}, \delta\right)$. Since in each sub-complex every other row is zero, differentials on odd pages are zero. Summing sub-complexes back together we obtain the result. □

Table 5.1 represents the first page of the spectral sequence from Corollary 5.18, i.e. $H(\mathrm{fGCc}_0^{\not\circlearrowleft}, \delta)$. The column number represents the degree, i.e. the number of edges e and the row number represents $b = e - v$, and the displayed numbers are the dimensions of the respective parts of the graph cohomology $H(\mathrm{fGCc}_0^{\geq 2\,\not\circlearrowleft}, \delta)$.[1]

[1] The numbers in the table have been partially taken from [3], and have partially been computed by the authors adviser T. Willwacher (unpublished). The latter computations have been performed in floating point arithmetic (due to limited computer power) and hence are not mathematically rigorous.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0		0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0
1			0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2				0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3					0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4						0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5							0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
6								0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
7									0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0
8										0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0
9											0	0	0	0	0	0	0	0	0	0	1	0	0	2	0	0	0	1
10												0	0	0	0	0	0	0	0	0	0	0	2	0	0	2	?	?
11													0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	?
12														0	0	0	0	0	0	0	0	0	0	0	0	0	3	?

Table 5.1: Table of even graph cohomology. The arrows indicate some cancellations in the spectral sequence from Corollary 5.18.

Remark 5.19. The first result in [7] gives cancellations of classes in the same table, but with another differential $(\delta + \nabla)$ and hence without the result that the differentials on odd pages are 0. Indeed, all pairs of cancelled classes are the same, so both tables are the same, and already with the differential $\delta + \nabla$ there were no cancellations on odd pages. However, in this thesis we work only with the differential $\tilde{\delta}$. Therefore the equality of tables is not needed, and we skip the too technical proof of it.

Chapter 6

Extra differential for graph complex, n odd

In this chapter we study the cohomology of $(\mathrm{fGCc}_n, \delta)$ for odd n. As a representative of odd n we work with the case $n = 1$. In this case the degree of a graph is the number of vertices minus 1. We introduce a changed differential on fGCc_1, leading to an improvement in understanding the standard cohomology. The result is from [7].

6.1 A new differential

We consider the following degree 1 element of fGCc_1:

$$m := \sum_{j\geq 2} \frac{1}{j!} \underset{\bullet}{\overset{\bullet}{\Big|}}\, {}^{j} = \sum_{j\geq 1} \frac{1}{(2j+1)!} \underset{\bullet}{\overset{\bullet}{\Big|}}\, {}^{2j+1} \tag{6.1}$$

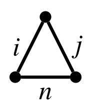
where the thick line with a number j represents j-fold edge, i.e. j edges connecting the same pair of vertices. Note that graphs with even-fold edges are zero because of the symmetry.

Lemma 6.1. *The element m is a Maurer-Cartan element, i.e. $\delta m + \frac{1}{2}[m, m] = 0$.*

Proof. Let

$$m' := m + \underset{\bullet}{\overset{\bullet}{\Big|}} = \sum_{j\geq 0} \frac{1}{(2j+1)!} \underset{\bullet}{\overset{\bullet}{\Big|}}\, {}^{2j+1} \tag{6.2}$$

It is enough to check that $m' \bullet m' = 0$. One checks that

$$\underset{\bullet}{\overset{\bullet}{\Big|}}\, {}^{m} \bullet \underset{\bullet}{\overset{\bullet}{\Big|}}\, {}^{n} = 2 \sum_{i+j=m} \binom{m}{i}$$ 

.

Let us denote the graphs on the left by X_m and X_n and the graph on the right by $Y_{i,j,n}$. By symmetry, we have the following identities

$$X_m = (-1)^{m+1} X_m \qquad\qquad Y_{i,j,n} = (-1)^{n+1} Y_{j,i,n} = (-1)^{i+1} Y_{i,n,j}.$$

In particular $X_m = 0$ for m even. One computes:

$$\begin{aligned} m' \bullet m' &= \sum_{k,l\geq 1, odd} \frac{1}{k!l!} X_k \bullet X_l = 2 \sum_{k,l\geq 1, odd} \sum_{i+j=k} \frac{1}{i!j!l!} Y_{i,j,l} \\ &= 4 \sum_{i\geq 0, even} \sum_{j,l\geq 1, odd} \frac{1}{i!j!l!} Y_{i,j,l} = 2 \sum_{i\geq 0, even} \sum_{j,l\geq 1, odd} \frac{1}{i!j!l!} (Y_{i,j,l} + Y_{i,l,j}) \\ &= 0. \end{aligned}$$

□

Hence, $\delta + [m, \cdot] = [m', \cdot]$ is another differential on fGC_1 and fGCc_1. It turns out that this cohomology can in fact be computed completely.

Theorem 6.2. *$H(\mathrm{fGCc}_1, \delta + [m, \cdot])$ is one dimensional, the class being represented by*

(6.3)
$$c := \sum_{j>0} \frac{j}{(2j+1)!}$$  $2j+1$.

The proof of this theorem will occupy the rest of this chapter. Let us however give a brief overview of the strategy, for the reader's convenience. There are four steps.

1. We introduce an auxiliary graph complex, the *dotted complex* DGC_1, which is larger than fGCc_1 in the sense that we allow another type of edges, "dotted" edges. We show in Proposition 6.5 and Proposition 6.9 that this larger complex is still essentially quasi-isomorphic to fGCc_1.

2. The (deformed) differential on the dotted complex contains a term which formally resembles the differential ∇ (i.e., it adds one dotted edge in all possible ways) discussed in the previous chapter for the even-n-version of the graph complexes. By a similar argument as that leading to the proof of Corollary 5.7, we reduce (in Proposition 6.14) the dotted graph complex to a much simpler complex, which we call waved complex.

3. The waved complex is shown to be acyclic in Proposition 6.12.

4. From this result we then finally deduce Theorem 6.2 in Section 6.5.

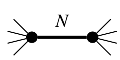
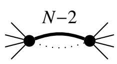
6.2 Dotted complex

Let DGC_1 be a complex similar to fGC_1, but with additional type of edge, the "dotted" edge •···• of degree 1 and parity $-$, i.e. graph with permuted dotted edges are identified with itself multiplied by sign of the permutation. Dotted tadpoles are not allowed by definition. On DGC_1 the Lie bracket is defined analogously to that on fGC_1. The standard differential on DGC_1 is $\delta = [$•—•$, \cdot]$, like on fGC_1. The strict definitions like those in Chapter 2 are left to the reader.

Definition 6.3. Let $\phi : \mathrm{DGC}_1 \to \mathrm{DGC}_1$ be defined on a graph Γ by

$$\phi(\Gamma) = (-1)^{v(\Gamma)} \sum_{e \in V(\Gamma)^2} \phi^e(\Gamma)$$

where $V(\Gamma)$ is the set of vertices of Γ, $v(\Gamma) = |V(\Gamma)|$ and for e being a pair of vertices with more than one standard edge between them

ϕ^e  $= N(N-1)$  ,

where standard edges are considered all to tend to the same direction. If there is a dotted edge between vertices of e, or there is no multiple edge, $\phi^e = 0$.

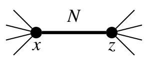
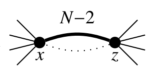
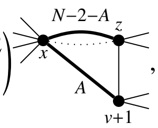
Proposition 6.4. *On* DGC_1 *it holds that*

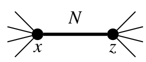
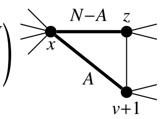
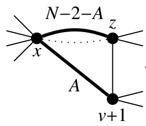
- $\phi^2 = 0$,
- $\delta\phi + \phi\delta = 0$,
- $(\delta + \phi)^2 = 0$.

Proof. Only the second claim is non-trivial.

$$\begin{aligned}(\delta\phi+\phi\delta)(\Gamma) &= \delta\left((-1)^v\sum_{x,y\in V}\phi^{(x,y)}(\Gamma)\right)+\phi\left(\sum_{z\in V}\left(\frac{1}{2}s_z(\Gamma)-a_z(\Gamma)\right)\right)=\\ &=(-1)^v\sum_{x,y,z\in V}\left(\frac{1}{2}s_z\phi^{(x,y)}(\Gamma)-a_z\phi^{(x,y)}(\Gamma)\right)+\\ &\quad+(-1)^{v+1}\left(\sum_{z\in V}\sum_{x,y\in V-\{z\}}\left(\frac{1}{2}\phi^{(x,y)}s_z(\Gamma)-\phi^{(x,y)}a_z(\Gamma)\right)+2\sum_{z\in V}\sum_{x\in V-\{z\}}\left(\phi^{(x,z)}s_z(\Gamma)-\phi^{(x,z)}a_z(\Gamma)\right)\right)=\\ &=(-1)^v 2\sum_{\substack{x,z\in V\\ x\neq z}}\left(\frac{1}{2}s_z\phi^{(x,z)}(\Gamma)-a_z\phi^{(x,z)}(\Gamma)-\phi^{(x,z)}s_z(\Gamma)+\phi^{(x,z)}a_z(\Gamma)\right)=\\ &=(-1)^v\sum_{\substack{x,z\in V\\ x\neq z}}\left(s_z\phi^{(x,z)}(\Gamma)-2\phi^{(x,z)}s_z(\Gamma)\right).\end{aligned}$$

It holds that

$s_z\phi^{(x,z)}$  $= N(N-1)s_z$  $= N(N-1)2\sum_{A=0}^{N-2}\binom{N-2}{A}$  ,

$\phi^{(x,z)}s_z$  $=\phi^{(x,z)}\sum_{A=0}^{N}\binom{N}{A}$  $=\sum_{A=0}^{N}\binom{N}{A}(N-A)(N-A-1)$  .

So $s_z\phi^{(x,z)}=2\phi^{(x,z)}s_z$, and the claimed relation follows. □

Since ϕ is of degree 1, the proposition implies that ϕ and $\delta+\phi$ are both differentials on DGC_1.

There is a natural projection $f\colon(\mathrm{DGC}_1,\delta+\phi)\to(\mathrm{fGC}_1,\delta)$ which sends a graph without dotted edges to itself, and a graph with at least one dotted edge to 0. Clearly, it is a morphism of complexes, and also of graded Lie algebras. Indeed we are interested in the connected part of the complexes, $\mathrm{fGCc}_1\subset\mathrm{fGC}_1$ and $\mathrm{DGCc}_1\subset\mathrm{DGC}_1$ defined in the usual way. The restriction $f\colon(\mathrm{DGCc}_1,\delta+\phi)\to(\mathrm{fGCc}_1,\delta)$ is well defined.

The subcomplex $\left(\mathrm{fGCc}_1^{\not\Leftarrow},\delta\right)\subset(\mathrm{fGCc}_1,\delta)$ generated by all graphs without multiple edges is also a subcomplex of $(\mathrm{DGCc}_1,\delta+\phi)$. We have the following proposition.

Proposition 6.5. *The inclusion* $\left(\mathrm{fGCc}_1^{\not\Leftarrow},\delta\right)\to(\mathrm{DGCc}_1,\delta+\phi)$ *is a quasi-isomorphism.*

Proof. On the mapping cone of the inclusion we set up a spectral sequence on the number of vertices. It clearly converges correctly.

The associated graded complex is the mapping cone of the inclusion $\left(\mathrm{fGCc}_1^{\not\Leftarrow},0\right)\to(\mathrm{DGCc}_1,\phi)$. The target complex is $(\mathrm{DGCc}_1,\phi)=(\mathrm{fGCc}_1^{\not\Leftarrow},0)\oplus(C,\phi)$ where $(C,\phi)\subset(\mathrm{DGCc}_1,\phi)$ is a sub-complex generated by graphs which have at least one dotted edge or double edge. Using the homotopy $h\colon C\to C$ which transforms the dotted edge to a double edge it can be seen that (C,ϕ) is acyclic, thus concluding the proof. □

We denote

(6.4) $p :=$ 

and let

$$\mathrm{DGC}_1^p:=\mathrm{DGC}_1\oplus\mathbb{K}p,\quad \mathrm{DGCc}_1^p:=\mathrm{DGCc}_1\oplus\mathbb{K}p. \tag{6.5}$$

The Lie bracket on DGC_1 naturally extends to DGC_1^p, the Lie bracket with p being the operation of adding one dotted edge, in all possible ways. The projection $f\colon\mathrm{DGC}_1\to\mathrm{fGC}_1$ extends to a function $f\colon\mathrm{DGC}_1^p\to\mathrm{fGC}_1$ by setting $f(p)=0$, and it is a map of complexes.

Corollary 6.6. *The projection* $f\colon \left(\mathrm{DGCc}_1^p, \delta+\phi\right) \to (\mathrm{fGCc}_1, \delta)$ *is a quasi-isomorphism.*

Proof. Let $\Theta :=$  and $\zeta :=$ . Let $\mathrm{fGCc}_1^{\not\leftrightarrow\,\Theta} := \mathrm{fGCc}_1^{\not\leftrightarrow} \oplus \mathbb{K}\Theta$ and $\mathrm{fGCc}_1^{\not\leftrightarrow\,\Theta p\zeta} := \mathrm{fGCc}_1^{\not\leftrightarrow} \oplus \mathbb{K}\Theta \oplus \mathbb{K}p \oplus \mathbb{K}\zeta$.

Proposition 6.5, with one extra class, ensures that the inclusion $\left(\mathrm{fGCc}_1^{\not\leftrightarrow\,\Theta p\zeta}, \delta+\phi\right) \to \left(\mathrm{DGCc}_1^p, \delta+\phi\right)$ is a quasi-isomorphism. Clearly, the projection $\left(\mathrm{fGCc}_1^{\not\leftrightarrow\,\Theta p\zeta}, \delta+\phi\right) \to \left(\mathrm{fGCc}_1^{\not\leftrightarrow\,\Theta}, \delta)\right)$ is also a quasi-isomorphism. Theorem 4.2 together with Proposition 2.22 implies that the inclusion $\left(\mathrm{fGCc}_1^{\not\leftrightarrow\,\Theta}, \delta\right) \to (\mathrm{fGCc}_1, \delta)$ is also a quasi-isomorphism. Composing all these maps on the level of homology leads to the result. □

6.3 Extra differential on the dotted complex

We consider the following degree 1 element of DGC_1^p:

$$\tilde{m} \;:=\; m \;+ p = \sum_{j\geq 1} \frac{1}{(2j+1)!}$$  $${}^{2j+1} \;+$$  .

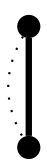
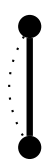
Lemma 6.7. *The element* $\tilde{m}$ *is a Maurer-Cartan element in* DGC_1^p*, i.e.*

$$(\delta+\phi)\tilde{m} + \frac{1}{2}[\tilde{m},\tilde{m}] = 0.$$

Proof. It holds that

$$\begin{aligned}(\delta+\phi)\tilde{m} + \frac{1}{2}[\tilde{m},\tilde{m}] &= \delta m + \frac{1}{2}[m,m] + \delta p + \phi m + \phi p + [m,p] + \frac{1}{2}[p,p] \\ &= \zeta + \phi m + [m,p] = \phi m + [m',p]\,.\end{aligned}$$

But now

$$\phi m = \sum_{j\geq 1} \frac{-2j(2j+1)}{(2j+1)!}$$  $${}^{2j-1} \;= -\sum_{j\geq 0} \frac{1}{(2j+1)!}$$  $${}^{2j+1} \;= -[m',p]$$

and hence the lemma is shown. □

The lemma implies that $\delta+\phi+[\tilde{m},\cdot] = \phi + [m'+p,\cdot]$ is another differential on DGC_1^p, called *twisted* differential. We have $f(\tilde{m}) = m$, so f is a map of complexes $(\mathrm{DGCc}_1^p, \delta+\phi+[\tilde{m},\cdot]) \to (\mathrm{fGCc}_1, \delta+[m,\cdot])$. We need a general statement.

Proposition 6.8. *Let* (C,d) *and* (D,d) *be differential graded Lie algebras equipped with descending complete filtrations* $C = \mathcal{F}^0C \supset \mathcal{F}^1C \supset \cdots$, $D = \mathcal{F}^0D \supset \mathcal{F}^1D \supset \cdots$. *We assume that these filtrations are compatible with the Lie structure in the sense that* $[\mathcal{F}^aC, \mathcal{F}^bC] \subset \mathcal{F}^{a+b}C$ *for every a and b, and that the spaces* $\mathcal{F}^aC/\mathcal{F}^{a+1}C$ *and* $\mathcal{F}^aD/\mathcal{F}^{a+1}D$ *are finite dimensional in each (cohomological) degree. Let* $m \in \mathcal{F}^1C$ *be Maurer-Cartan element, and let* $f\colon C \to D$ *be a morphism of differential graded Lie algebras respecting the filtrations, and inducing a quasi-isomorphism on the associated graded complexes. Then* $f(m) \in \mathcal{F}^1D$ *is Maurer-Cartan element in D and f induces a quasi-isomorphism of the twisted dg Lie algebras* $(C, d+[m,\cdot]) \to (D, d+[f(m),\cdot])$.

Proof. The map f clearly commutes with new differentials $d+[m,\cdot]$ and $d+[f(m),\cdot]$. On the mapping cone of $f : (C, d+[m,\cdot]) \to (D, d+[f(m),\cdot])$ we set up a spectral sequence arising from the filtrations given. Using the finite-dimensionality Proposition B.2 implies that the spectral sequence converges correctly. Since $m \in \mathcal{F}^1C$, $[m,\cdot]$ and $[f(m),\cdot]$ are not part of the first differential, and the associated graded complex is the same as of $f\colon C \to D$, hence acyclic. Hence the result. □

Proposition 6.9. *The projection* $f\colon (\mathrm{DGCc}_1^p, \delta+\phi+[\tilde{m},\cdot]) \to (\mathrm{fGCc}_1, \delta+[m,\cdot])$ *is a quasi-isomorphism.*

Proof. We define the "Lie degree" of a graph to be the number of edges plus twice the number of dotted edges minus the number of vertices, and set up a spectral sequence on Lie degree. The finite dimensionality condition in Proposition 6.8 is easily checked, and using Corollary 6.6, the Proposition implies the result. □

6.4 Waved complex

Let WGC_1 be a graph complex generated by graphs of vertices of degree 1 and indistinguishable directed edges $\bullet\!\rightsquigarrow\!\bullet$ of degree 0. The direction of the edge can not be reversed, and every pair of vertices is connected with exactly one edge, of some orientation. In other words, disregarding the edge orientations all the graphs are full graphs. The strict definitions like those in Chapter 2 are left to the reader.

Definition 6.10. Let $d_1 : \mathrm{WGC}_1 \to \mathrm{WGC}_1$ be defined on a graph Γ with v vertices by

$$d_1(\Gamma) = \sum_{x\in V(\Gamma)} d_x(\Gamma) \tag{6.6}$$

where d_x adds another vertex $v+1$, connects $\underset{x}{\bullet}\!\rightsquigarrow\!\underset{v+1}{\bullet}$ and connects every other vertex of Γ to the new vertex with the edge of the same direction as the edge it connects to x.

Let $d_{\mathrm{in}} : \mathrm{WGC}_1 \to \mathrm{WGC}_1$ be defined on the graph Γ by adding the new vertex $v+1$ and connecting every other vertex to it with an edge directed towards it, and d_{out} does the same with the opposite direction.

Let

$$d = d_1 - d_{\mathrm{in}} + d_{\mathrm{out}}. \tag{6.7}$$

Proposition 6.11. *On* DGC_1 *it holds that*

- $d_1^2 = 0$,
- $d^2 = 0$.

Proof. Easy verification. □

Maps d_1 and d are of degree 1, so the proposition implies that they are both differentials on WGC_1.

Proposition 6.12. *The complex* (WGC_1, d) *is acyclic.*

Proof. We look for a homotopy $h\colon \mathrm{WGC}_1 \to \mathrm{WGC}_1$ of degree -1 such that $hd + dh = -\mathrm{Id}$.

Let *the maximum* vertex in the graph Γ be the vertex for which all adjacent edges head towards it, if such a vertex exists. Let the map h delete the maximum vertex of a graph Γ and its adjacent edges if such a vertex exists, and set $h(\Gamma) = 0$ otherwise. The maps d_1 and d_{out} can not create or remove the maximum vertex, so if there is no such vertex in Γ then $(hd + dh)(\Gamma) = -hd_{\mathrm{in}}(\Gamma) = -\Gamma$. On the other hand, If x is the maximum vertex, then $(hd + dh)(\Gamma) = (hd_x - hd_{\mathrm{in}} - d_{\mathrm{in}}h)(\Gamma) = -\Gamma$. □

We construct a map $g\colon \mathrm{WGC}_1 \to \mathrm{DGC}_1$ which maps a graph to the graph with the same vertices, and edges $\underset{x}{\bullet}\!\rightsquigarrow\!\underset{y}{\bullet}$ are replaced with

$$\sum_{j\geq 0} \frac{1}{j!}\ \underset{x}{\bullet}\!\xrightarrow{\ j\ }\!\underset{y}{\bullet}$$

where the direction of all edges is the same as the direction of the former waved edge (from x to y).

Proposition 6.13. *The map* $g : (\mathrm{WGC}_1, d) \to (\mathrm{DGC}_1, \delta + \phi + [\tilde{m}, \cdot])$ *is a map of complexes, i.e.*

$$g(d\Gamma) = (\delta + \phi)g(\Gamma) + [\tilde{m}, g(\Gamma)]\,.$$

Proof. The part $\phi + [p, \cdot]$ does not change the number of vertices, and acts on every pair of them separately. It is easy to see that it sends the image of g to 0. Therefore it is enough to prove that

$$g(d\Gamma) = \delta g(\Gamma) + [m, g(\Gamma)] = [m', g(\Gamma)]\,.$$

Note that

$$m' = \sum_{j\geq 0} \frac{1}{(2j+1)!}\ \bullet\!\overset{2j+1}{\text{---}}\!\bullet\ = \sum_{j\geq 0} \frac{1}{j!}\ \bullet\!\overset{j}{\text{---}}\!\bullet$$

because the additional terms are 0 by symmetry reasons. Therefore

$$\begin{aligned}
[m', g(\Gamma)] &= m' \bullet g(\Gamma) - (-1)^{|\Gamma|} g(\Gamma) \bullet m' = \\
&= \sum_{j\geq 0} \frac{1}{j!} \bullet\!\overset{j}{\text{—}}\!(g\Gamma) - \sum_{j\geq 0} \frac{1}{j!} (g\Gamma)\!\overset{j}{\text{—}}\!\bullet - (-1)^{|\Gamma|} \sum_{x\in V(\Gamma)} \frac{1}{j!}\; g\Gamma = \underset{x}{\left(\,\overset{\bullet}{\underset{\bullet}{|}}\, j\right)} = \\
&= g\left(\bullet\!\rightsquigarrow\!(\Gamma)\right) - g\left((\Gamma)\!\rightsquigarrow\!\bullet\right) - (-1)^{|\Gamma|} g\left(\sum_{x\in V(\Gamma)} \Gamma = \underset{x}{\left(\overset{\bullet}{\underset{\bullet}{\wr}}\right)}\right) = \\
&= g\left(d_{\mathrm{out}}\Gamma\right) - g\left(d_{\mathrm{in}}\Gamma\right) + g\left(\sum_{x\in V(\Gamma)} d_x(\Gamma)\right) = g(d\Gamma).
\end{aligned}$$

Here in the third line we represented the various terms of the differential d on WGC_1 by little pictograms, which we hope the reader finds self-explanatory. □

Proposition 6.14. *The map* $g\colon (\mathrm{WGC}_1, d) \to (\mathrm{DGC}_1, \delta + \phi + [\tilde{m}, \cdot])$ *is a quasi-isomorphism.*

Proof. On the mapping cone we set up a spectral sequence on the number of vertices. The spectral sequence clearly converges correctly. On the first page we have the mapping cone of $g : (\mathrm{WGC}_1, 0) \to (\mathrm{DGC}_1, \phi + [p, \cdot])$.

Now, as in (A.4) we can distinguish vertices for every particular number of them, and look at the mapping cone of $g : (\bar{\mathrm{V}}_v\mathrm{WGC}_1, 0) \to (\bar{\mathrm{V}}_v\mathrm{DGC}_1, \phi + [p, \cdot])$. The differential $\phi + [p, \cdot]$ acts on every pair of vertices (x, y) separately, so the mapping cone splits as a tensor product of complexes for every pair (x, y).

An easy investigation of the action of $\phi+[p, \cdot]$ on the pair (x, y) leads to the 2 dimensional homology generated by $\sum_{j \text{ even}} \frac{1}{j!} \underset{x}{\bullet}\!\overset{j}{\text{—}}\!\underset{y}{\bullet}$ and $\sum_{j \text{ odd}} \frac{1}{j!} \underset{x}{\bullet}\!\overset{j}{\text{—}}\!\underset{y}{\bullet}$. A different basis is $\left\{ g\left(\underset{x}{\bullet}\!\rightsquigarrow\!\underset{y}{\bullet}\right), g\left(\underset{x}{\bullet}\!\leftrightsquigarrow\!\underset{y}{\bullet}\right)\right\}$. Now it is easy to see that g on the second page is actually an isomorphism. Therefore, the mapping cone on this page is acyclic, and so is the starting mapping cone. □

6.5 The end of the proof

Now we can obtain the cohomology of the complex fGC_1 with respect to the differential $\delta + [m, \cdot]$.

Proof of Theorem 6.2. Propositions 6.12 and 6.14 imply that $(\mathrm{DGC}_1, \delta + \phi + [\tilde{m}, \cdot])$ is acyclic.

We claim that the connected part $(\mathrm{DGCc}_1, \delta + \phi + [\tilde{m}, \cdot])$ is also acyclic. Suppose the opposite, and let the first class in cohomology appear in degree t, i.e. the combined number of vertices and dotted edges is t. Clearly $t \geq 2$.

We set up a spectral sequence on $(\mathrm{DGC}_1, \delta+\phi+[\tilde{m}, \cdot])$ on the number of connected components. It clearly converges correctly. Since the differential on the associated graded respects the numbers of connected components by construction, on the first page of the spectral sequence we find the symmetric product of $H(\mathrm{DGCc}_1, \delta + \phi + [\tilde{m}, \cdot])$.

The subsequent differentials in the spectral sequence will (i) reduce the numbers of connected components and (ii) increase the degree by 1. By our (contrapositive) assumption we have a lowest degree connected cohomology class in degree $t \geq 2$. But then the higher symmetric powers $S^k(H(\mathrm{DGCc}_1, \delta + \phi + [\tilde{m}, \cdot]))$ will be concentrated in degrees $kt \geq 2t$. Hence, our degree t connected class cannot be cancelled on further pages of the spectral sequence, contradicting the acyclicity of $(\mathrm{DGC}_1, \delta + \phi + [\tilde{m}, \cdot])$. Hence we conclude that $H(\mathrm{DGCc}_1, \delta + \phi + [\tilde{m}, \cdot]) = 0$.

Adding p to $(\mathrm{DGCc}_1, \delta + \phi + [\tilde{m}, \cdot])$ creates one class in cohomology. It holds that

$$\tilde{\delta} p + [\tilde{m}, p] = \sum_{j\geq 0} \frac{1}{(2j+1)!} \overset{\bullet}{\underset{\bullet}{\Big)}}{}^{2j+1} =: b.$$

The differential maps it to 0, and since $(\mathrm{DGCc}_1, \tilde{\delta} + [\tilde{m}, \cdot])$ is acyclic there is $c \in \mathrm{DGCc}_1$ such that $b = \tilde{\delta} c + [\tilde{m}, c]$, so $p - c$ generates the only class of $(\mathrm{DGCc}_1^p, \tilde{\delta} + [\tilde{m}, \cdot])$. By induction it can be seen that

$$c = \sum_{j>0} \frac{j}{(2j+1)!} \overset{\bullet}{\underset{\bullet}{\Big|}}{}^{2j+1}.$$

Proposition 6.9 now implies that $H(\mathrm{fGCc}_1, \delta + [m, \cdot])$ is one dimensional with the class being represented by $f(p - c) = -c$. □

Remark 6.15. The proof of Theorem 6.2 we gave above is elementary and combinatorial. The more conceptual proof is sketched in [7, Appendix A].

6.6 A picture of the odd graph cohomology

Similarly as in odd case, there is a Corollary.

Corollary 6.16. *There is a spectral sequence converging to*

$$H(\mathrm{fGCc}_1, \delta + [m, \cdot]) = \mathbb{K}c$$

whose first page is

$$H(\mathrm{fGCc}_1, \delta).$$

Furthermore, all differentials on odd pages are 0*.*

Proof. We set the spectral sequence on $(\mathrm{fGCc}_1, \delta + [m, \cdot])$ on the number $b = e - v$. Clearly, on the first page we have $H(\mathrm{fGCc}_1, \delta)$.

Like in the odd case, the complex $(\mathrm{fGCc}_1, \delta + [m, \cdot])$ is in fact the direct sum of two complexes, namely its subcomplexes corresponding to even and odd b. Therefore, there can not be a non-zero differential changing the parity of b on any page of spectral sequence, and hence all differentials on odd pages are 0.

The subspace of fGCc_1 corresponding to a fixed number of vertices (degree) and a fixed b is finite-dimensional, so Proposition B.2 implies that spectral sequence converges correctly. □

Table 6.1 represents the second page of the spectral sequence from Corollary 6.16, i.e. $H(\mathrm{fGC}_1, \delta)$, where the column number represents the degree $d = v - 1$ and the row number represents the $b = e - v$. The numbers in the table represent the dimension of the respective subspace of $H(\mathrm{fGC}_1, \delta)$.[1] Some cancellations are represented by arrows in the table. The arrows with a question mark are conjectural, since the computed data is not sufficient to rule out cancellations with classes in the unknown region.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	1	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	1	0	0	3	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	2	0	0	4	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	?	?	?	?	?	5	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	?	?	?	?	?	?	?	6	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	?	?	?	?	?	?	?	?	8	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	?	?	?	?	?	?	?	?	?	9

Table 6.1: The table of cohomology of fGCc_1. The arrows indicate some cancellations in the spectral sequence.

[1] The numbers in the table have been partially taken from [3], and have partially been computed by the author's adviser T. Willwacher (unpublished).

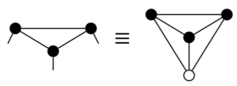
Chapter 7

Extra differential for hairy graph complex, m even

In this chapter we study the cohomology of $(\mathrm{HGC}_{m,n}, \delta)$ for even m. As a representative of even m we work with the case $m = 0$. We introduce a changed differential on $\mathrm{HGC}_{0,n}$, leading to an improvement in understanding the standard cohomology. The result is from [8].

7.1 A new differential

If $m = 0$ hairs are of the same degree and parity as edges. Therefore, they can be understood as edges going towards a special vertex.



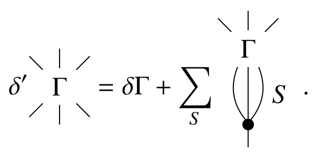
The standard differential δ splits ordinary vertices. It is natural to extend the differential to splitting all vertices.

Definition 7.1. Let $\delta' : \mathrm{HGC}_{0,n} \to \mathrm{HGC}_{0,n}$ be defined on a graph Γ by

$$\delta'(\Gamma) = \sum_{x\in V(\Gamma)} \left(\frac{1}{2} s_x(\Gamma) - a_x(\Gamma)\right) + s_{ext}(\Gamma) - a_{ext}(\Gamma),$$

where s_{ext} splits the special vertex, i.e. inserts $\circ\!\!-\!\!\bullet$ instead of the special vertex and sums over all possible ways of connecting edges that have been connected to the special vertex (hairs) to new special and new standard vertex, and a_{ext} adds an edge at the special vertex, i.e. adds $\circ\!\!-\!\!\bullet$ on the special vertex, while all edges (hairs) stay on it.

An equivalent definition, without mentioning special vertex, is as follows:



Here δ is the original differential and the sum on the right-hand side goes over all subsets S of the set of hairs with at least two elements.

Proposition 7.2. *On $\mathrm{HGC}_{0,n}$ it holds that $\delta'^2 = 0$, i.e. it is a differential.*

Proof. Easy verification. □

Theorem 7.3 ([8, Proposition 1], c.f. [15, Appendix B])**.** *The complex $(\mathrm{HGC}_{0,n}, \delta')$ is acyclic. The spectral sequence on the number of hairs, that has the standard differential as the first differential, converges correctly.*

Proof. Let for a graph $\Gamma \in \mathrm{HGC}_{0,n}$

$$t(\Gamma) = \begin{cases} 1 & \Gamma \text{ has 1 hair,} \\ 0 & \Gamma \text{ has more hairs,} \end{cases}$$

and let us set up a spectral sequence on degree minus t. The first differential is the term that produces one hair from the graph that has more hairs. There is a homotopy that deletes the hairy vertex and transforms its edges to hairs, leading to the first result.

The differential δ' does not change $f = e + h - v$, so the complex split as

$$(\mathrm{HGC}_{0,n}, \delta') = \prod_{f \in \mathbb{Z}} \left(\mathrm{F}_f \mathrm{HGC}_{0,n}, \delta' \right). \tag{7.1}$$

In the spectral sequence on the number of hairs h, for fixed f, row h and degree, all numbers v, e and h are fixed, so the space is finite-dimensional. Therefore Corollary B.3 together with Proposition B.2 implies that the spectral sequence converges correctly. □

7.2 A Picture of the hairy graph cohomology: The waterfall mechanism

Realize that we have two extra differentials on hairy graph complex that make it (almost) acyclic (or with cohomology equal to assumed to be known non-hairy cohomology) and that make correctly converging spectral sequence with the standard differential as the first one. We call the spectral sequence arising from the deformed differential δ' of Theorem 7.3 the *first* spectral sequence, and the one arising from $\delta + \chi$ or $\delta + [\omega, \cdot]$ of Theorem 2.31, respectively 2.34, the *second*. Note that the first spectral sequence does not contain the hairless part, while the second one does for m and n of the same parity. The different constraint on valences does not matter because of Corollary 2.27.

Let us now describe how to construct from the above two spectral sequences a large set of additional non-trivial hairy graph cohomology classes by a process we call the waterfall mechanism.

Let us focus on the case of m, n even for concreteness, say $m = n = 0$. The case of m even and n odd, and the cases m odd, n even or odd in the later chapters, are treated analogously.

Note that the convergence of the spectral sequences imply that the hairy graph cohomology classes must come "in pairs". More concretely, given a hairy graph cohomology class Γ, it will survive up to some page of the spectral sequence, on which it is either killed by or kills (the image of) another hairy graph cohomology class.

More concretely, from Theorem 7.3 we see that if Γ lives in tri-degree

$$(\text{cohom. degree, number of hairs, loop order}) =: (d, h, b),$$

then the "partner class" that it kills (or is killed by) must live in tri-degree $(d + 1, h - j, b + j)$ or $(d - 1, h + j, b - j)$ for some yet unknown positive integer j.

Hence from the existence of the non-trivial class $\Gamma \in \mathrm{HGC}_{0,0}$ we can conclude that there is another nontrivial class in $\mathrm{HGC}_{0,0}$ whose tri-degree lies on a union of half-lines in $\mathbb{Z}^3$. A representative of this (or rather, some such) class may be constructed by following the spectral sequence.

Now consider the second spectral sequence arising from the deformed differential $\delta + \chi$ of Theorem 2.31. Note that here h can be zero, and the cohomology in that case is the cohomology $H(\mathrm{GC}_0, \delta)$. As before, nontrivial hairy graph cohomology classes must kill or be killed by other non-trivial classes on some page of the spectral sequence. For this second spectral sequence, one can see that the partner class of a class in tri-degree (d, h, b) must live in tri-degree $(d + 1, h + j, b)$ or $(d - 1, h - j, b)$ for a positive integer j. In fact, it is shown in [17] that the spectral sequence abuts on the second page and hence $j = 1$.

Now, using the constraints provided by the first and the second spectral sequences together, we may construct a large set of hairy graph cohomology classes from (assumed to be known) non-hairy classes. Concretely, consider a non-hairy class $\Gamma \in H(\mathrm{GC}_0, \delta)$. It must kill (the image of) some hairy class Γ_1 in the second spectral sequence (in fact, it has to be $\Gamma_1 = \chi(\Gamma) \in \mathrm{H}_1\mathrm{HGC}_{0,0}$). Now, Γ_1 must be killed by (the image of) some other class, say Γ_2, in the first spectral sequence. The class Γ_2 must necessarily have more than one hair. Hence it must kill or be killed by (the image of) some other class in the second spectral sequence. This class must again be killed by (the image of) some class in the first spectral sequence etc., until at some point we reach another class in $H(\mathrm{GC}_0)$ where the first spectral sequence is not defined.

By this process we conclude from the existence of a non-hairy graph cohomology class the existence of a string of hairy graph cohomology classes. For an illustration of the process, see the computer generated table of the hairy graph cohomology in Figure 7.1, in which (some of) the cancellations in the two spectral sequences have been inscribed, for m even and for n both even and odd. Note that actual m in the figure is different than what we studied, but of the same parity, leading only to some degree shifting. For n even the figure does not contain non-hairy part, and starts with its hairy image $\Gamma_1 = \chi(\Gamma)$, the same as for n odd. We call the above mechanism to construct strings of hairy classes from non-hairy the "waterfall mechanism", by visual similarity of the cancellation pattern to a waterfall.

	1	2	3	4	5	6	7	8
9	1_{16}	1_{16}						
8		1_{15}						
7	1_{12}	1_{12}						
6		1_{11}						
5	1_8			1_{11}				
4				1_8				
3	1_4			1_7				
2								
1			1_1		1_1	1_4	1_1	$1_1, 1_4$

	1	2	3	4	5	6	7	8
9		1_6						
8								
7	1_5	1_5	2_5					
6			1_2					
5		1_2	2_2	3_2	5_2			
4			1_1	1_1	1_{-1}			
3	1_1	1_1	2_1	2_1	3_1			
2					1_{-2}	1_{-2}		
1			1_{-2}	1_{-2}	2_{-2}	$2_{-2}, 1_{-5}$	1_{-5}	

Figure 7.1: Computer generated table of the dimensions of the hairy graph cohomology $\dim H(\mathrm{HGC}_{2,2})$ (left) and $\dim H(\mathrm{HGC}_{2,3})$ (right). The rows indicate the number of hairs (↑), the columns the loop order (→). A table entry 1_3 means that there the degree 3 subspace is one-dimensional. The arrows indicate how classes kill each other in the first spectral sequences (solid) and the second spectral sequence (dotted). Not all cancellations are shown for the sake of readability. The computer program used approximate (floating point) arithmetic, so the displayed numbers should not be considered as rigorous results.

We note that in the case of even n the waterfall mechanism accounts for all hairy classes in the computer accessible regime. For odd n, we see additional 4 classes which do not originate from bald classes in hairs/loop order combinations $(1, 5)$, $(4, 5)$, $(4, 3)$ and $(6, 3)$.

Note also that the convergence of the spectral sequences to known data is not enough to reconstruct the precise location (i.e., tri-degree) of non-trivial cohomology classes in the "string", because we do not know in general on which pages certain classes are cancelled. We know from [17] that for n even all cancellations of the second spectral sequence happen on the second page, in consistency with the numerical tables. The picture in Figure 7.1 seems to indicate that other cancellations follow a fairly regular pattern. The study of the abutment properties might hence be an interesting topic of future work.

Chapter 8

Extra differential for hairy graph complex, m odd, general setting

In the last three chapters we study an extra differential on hairy graph complex $\mathrm{HGC}_{m,n}$ for m odd. The proof is a bit lengthy, and therefore we split it into two cases, for n odd and n even in next two chapters. In this chapter we introduce the differential and prepare some general results needed for both cases. As a representatives m odd and n odd and even we work with the complexes $\mathrm{HGC}_{-1,1}$ and $\mathrm{HGC}_{-1,0}$. In $\mathrm{HGC}_{-1,1}$ there are no tadpoles because of the symmetry, and in the other case we will also work with the complex without tadpoles $\mathrm{HGC}^{\not\circlearrowleft}_{-1,0}$. The result is from [23].

8.1 A new differential

Definition 8.1. Let $\Delta : \mathrm{fHGC}^{\not\circlearrowleft}_{-1,n} \to \mathrm{fHGC}^{\not\circlearrowleft}_{-1,n}$ be defined on a graph Γ by

$$\Delta\Gamma = \sum_{k\in H(\Gamma)} \sum_{x\in V(\Gamma)} \Delta_{k,x},$$

where $\Delta_{k,x}$ deletes the hair k on, say, the vertex y, and connects x to y if $x \neq y$, or is zero if $x = y$. To precisely define the sign of the resulting graph, we set that, before the acting of symmetric groups, a hair is considered to have the last number and a new edge get the next free number.

Proposition 8.2. *On* $\mathrm{fHGC}^{\not\circlearrowleft}_{-1,n}$ *it holds that*

- $\Delta^2 = 0$,
- $\delta\Delta + \Delta\delta = 0$,
- $(\delta + \Delta)^2 = 0$.

Proof. Easy verification. □

Since Δ is of degree 1 for all n, the proposition implies that Δ and $\delta + \Delta$ are differentials on $\mathrm{fHGC}^{\not\circlearrowleft}_{-1,n}$.

Technically, $\mathrm{H}_{\geq 1}\mathrm{fHGC}^{\not\circlearrowleft}_{-1,n}$ is not closed under the differentials Δ and $\delta + \Delta$. But, by the abuse of notation we will talk about the complexes $\left(\mathrm{H}_{\geq 1}\mathrm{fHGC}^{\not\circlearrowleft}_{-1,n}, \Delta\right)$ and $\left(\mathrm{H}_{\geq 1}\mathrm{fHGC}^{\not\circlearrowleft}_{-1,n}, \Delta + \delta\right)$, meaning the quotients $\left(\mathrm{fHGC}^{\not\circlearrowleft}_{-1,n}, \Delta\right) / \left(\mathrm{H}_0\mathrm{fHGC}^{\not\circlearrowleft}_{-1,n}, 0\right)$ and $\left(\mathrm{fHGC}^{\not\circlearrowleft}_{-1,n}, \Delta + \delta\right) / \left(\mathrm{H}_0\mathrm{fHGC}^{\not\circlearrowleft}_{-1,n}, \delta\right)$ respectively. Intuitively, it means identified hairless graphs with zero.

The main theorem we will prove in the last three sections is the following.

Theorem 8.3.

- *The complex* $(\mathrm{HGC}_{-1,1}, \delta + \Delta)$ *is acyclic.*

- *The cohomology* $H\left(\mathrm{HGC}^{\maltese}_{-1,0}, \delta + \Delta\right)$ *is one-dimensional, the class being represented by a graph with one vertex and three hairs on it.*
- *The spectral sequence on* $\left(\mathrm{HGC}^{\maltese}_{-1,n}, \delta + \Delta\right)$ *on the number of hairs, that has the standard differential as the first differential, converges correctly.*

The last claim of the Theorem is straightforward. We prove the first two claims in the following chapters, 9 and 10.

8.2 Deleting vertices in hairy graphs

In this section we introduce some maps on $\mathrm{fHGC}^{\maltese}_{-1,0}$ and $\mathrm{fHGC}^{\maltese}_{-1,1}$ that are generally deleting hairy vertices, and that will be used later in the proof of Theorem 8.3.

Definition 8.4. Let the operator of *deleting the hairy vertex* $D^{(1)} : \mathrm{H}_1\mathrm{fHGC}^{\maltese}_{-1,n} \to \mathrm{H}_0\mathrm{fHGC}^{\maltese}_{-1,n}$ be defined on graph Γ as

$$D^{(1)}(\Gamma) = (-1)^{v(x)}\tilde{D}_x$$

where x is the vertex with the hair, $v(x)$ is its valence and $\tilde{D}_x$ deletes vertex x and the hair, and sums over all ways of reconnecting edges that were connected to x to the other vertices, skipping graphs with a tadpole.

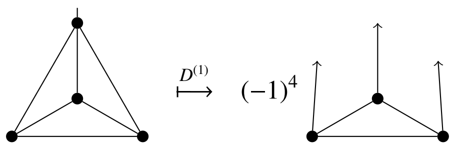


Figure 8.1: Example of the action $D^{(1)}$, deleting the hairy vertex. The graph at the right means the sum over all graphs that can be formed by attaching ends of arrows to vertices, skipping graphs with a tadpole.

Definition 8.5. Let $\tilde{D}^{(p)} : \mathrm{H}_1\mathrm{fHGC}^{\maltese}_{-1,n} \to \mathrm{H}_1\mathrm{fHGC}^{\maltese}_{-1,n}$ be defined on a graph Γ as

$$\tilde{D}^{(p)}(\Gamma) = (-1)^{v(x)} \sum_y v(x,y)\tilde{D}^y_x(\Gamma) \tag{8.1}$$

where x is the vertex with the hair, y runs through all vertices of Γ, $v(x,y)$ is the number of edges between vertices x and y, and whenever $v(x,y) > 0$ $\tilde{D}^y_x$ deletes vertex x, the hair and one edge between x and y, sums over all ways of reconnecting the other edges that were connected to x to the other vertices, skipping graphs with a tadpole, and adds a hair on y.

Let the operator of *pushing the hair* $D^{(p)} : \mathrm{H}_1\mathrm{fHGC}^{\geq 2\,\maltese}_{-1,n} \to \mathrm{H}_1\mathrm{fHGC}^{\geq 2\,\maltese}_{-1,n}$ be defined on graph Γ as

$$D^{(p)}(\Gamma) := \frac{1}{v(x)-1}\tilde{D}^{(p)}(\Gamma), \tag{8.2}$$

where x is the hairy vertex.

Lemma 8.6. *On* $\mathrm{H}_1\mathrm{fHGC}^{\geq 2\,\maltese}_{-1,n}$ *it holds that*

$$D^{(1)} = \Delta D^{(p)}.$$

Proof. Clear. □

Lemma 8.7. *On* $\mathrm{H}_1\mathrm{fHGC}^{\ddagger\,\maltese}_{-1,0}$ *it holds that*

$$D^{(1)}D^{(p)} = 0.$$

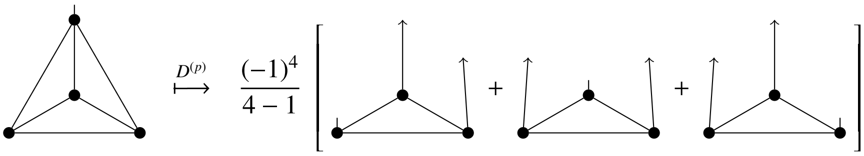


Figure 8.2: Example of the action $D^{(p)}$, pushing the hair. The graphs at the right mean the sum over all graphs that can be formed by attaching ends of arrows to vertices, without making a tadpole.

Proof. Recall that in $\mathrm{fHGC}^{\clubsuit}_{-1,0}$ there are no multiple edges. Let $\Gamma \in \mathrm{H}_1\mathrm{fHGC}^{\ddagger\clubsuit}_{-1,n}$ be a graph, and x the hairy vertex in Γ. One term in $D^{(1)}D^{(p)}(\Gamma)$ will delete the hair, the vertex x, one of its neighbours y and edge between them, and reconnect all edges that were connected to x and y elsewhere.

Recall that the constraint $\ddagger$ means that all vertices are at least 2-valent, and all hairy vertices are at least 3-valent. Therefore x has at least two edges adjacent to it. Chose an edge e at x that did not go towards y. That edge can go directly to its final destination, or first to y and then to final destination. Those two terms will cancel, implying the result. □

Proposition 8.8. *In* $\mathrm{H}_1\mathrm{fHGC}^{\clubsuit}_{-1,n}$ *it holds that*

$$\delta D^{(1)} - D^{(1)}\delta = \Delta.$$

Proof. Let $\Gamma \in \mathrm{H}_1\mathrm{fHGC}^{\clubsuit}_{-1,n}$ be a graph and let x be the hairy vertex in Γ.

$$\begin{aligned}
&\delta D^{(1)}(\Gamma) - D^{(1)}\delta(\Gamma) = \\
&\quad = \sum_{\substack{y\\ y\neq x}} \left(\frac{1}{2}s_y D_x(\Gamma) - a_y D_x(\Gamma)\right) - D_x \sum_{\substack{y\\ y\neq x}} \left(\frac{1}{2}s_y(\Gamma) - a_y(\Gamma)\right) - D_x\left(s'_x(\Gamma) - a_x(\Gamma)\right) + D_z e(\Gamma) = \\
&\quad = \frac{1}{2}\sum_{\substack{y\\ x\neq y}} s_y D_x(\Gamma) - \sum_{\substack{y\\ x\neq y}} a_y D_x(\Gamma) - \frac{1}{2}\sum_{\substack{y\\ x\neq y}} D_x s_y(\Gamma) + \sum_{\substack{y\\ x\neq y}} D_x a_y(\Gamma) - D_z s'_x(\Gamma) + D_x a_x(\Gamma) + D_z e(\Gamma),
\end{aligned}$$

where y runs through $V(\Gamma)$, D_x deleted the vertex x and a hair on it, e is extracting the hair, z is the name of the newly added vertex and s'_x is the part of s_x which moves the hair to the new vertex z. It holds $s_x = 2s'_x$.

Using the same arguments as in the proof of Proposition 5.12 it follows that:

$$s_y D_x(\Gamma) = D_x s_y(\Gamma),$$

$$D_z s'_x(\Gamma) = -\sum_{\substack{y\\ y\neq x}} a_y D_x(\Gamma),$$

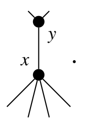
$$D_x a_x(\Gamma) = -\sum_{\substack{y\\ y\neq x}} D_x a_y(\Gamma).$$

It clearly holds that

$$D_z e(\Gamma) = \Delta(\Gamma),$$

so the claimed formula follows. □

The following map makes sense only for hairs of parity +, i.e. even n, because only in that case there can be more than one hair on the same vertex. Recall that a flower on vertex y with a root x is a sub-graph consisting of a trivalent vertex y that has two hairs, its hairs, edge and another end x of the edge, like in the diagram:



Definition 8.9. Let the operator of *deleting the flower* $D^{(2)} : \mathrm{H}_2\mathrm{fHGC}^{✿}_{-1,0} \to \mathrm{H}_0\mathrm{fHGC}^{✿}_{-1,0}$ be defined on a graph Γ with a flower on vertex y with a root x to be

$$D^{(2)}(\Gamma) = (-1)^{v(x)}\tilde{D}_{x,y}$$

where $v(x)$ is the valence of x, $\tilde{D}_{x,y}$ deletes a flower, i.e. vertices x and y, the edge between them and both hairs, and sums over all ways of reconnecting other edges that were connected to x to the other vertices, skipping graphs with a tadpole. For the matter of sign we consider the edge in the flower to be the last one.

If Γ does not have a flower, $D^{(2)}(\Gamma) = 0$.

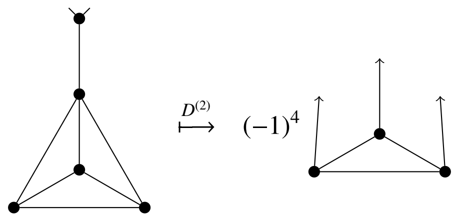


Figure 8.3: Example of the action $D^{(2)}$, deleting the flower. The graph at the right means the sum over all graphs that can be formed by attaching ends of arrows to vertices, without making a tadpole.

Proposition 8.10.

- *In the connected part* $\mathrm{H}_2\mathrm{fHGCc}^{✿}_{-1,0}$ *it holds that*

$$D^{(1)}\Delta = 2\nabla\left(\delta D^{(2)} + D^{(2)}\delta\right).$$

- *In* $\mathrm{H}_2\mathrm{fHGC}_{-1,1}$ *it holds that*

$$D^{(1)}\Delta = 0.$$

Proof. We do the proof of the first claim for four different cases of a graph $\Gamma \in \mathrm{H}_2\mathrm{fHGCc}^{✿}_{-1,0}$:

1. Γ has hairs on two different vertices, x and y. By the definition $\delta D^{(2)}(\Gamma) = \delta(0) = 0$. Differential δ can not move hairs to the same vertex, so it is also $D^{(2)}\delta(\Gamma) = 0$. The left-hand side is

$$D^{(1)}\Delta(\Gamma) = D^{(1)}(\Delta_x(\Gamma) + \Delta_y(\Gamma)) = D_y\Delta_x(\Gamma) + D_x\Delta_y(\Gamma),$$

where Δ_x and Δ_y are parts of Δ that act on the hair at x, respectively at y. In the first term Δ_x first deletes the hair on x and connects an edge f from x to all other vertices. Then D_y deletes a vertex y and reconnects all its edges to other vertices in all possible ways. If f has been connected to y, it is also reconnected to all other vertices, what is exactly the same term as if f has been connected at first to its final destination, but with the opposite sign. So, the terms cancel and $D_y\Delta_x(\Gamma) = 0$. The same argument leads to $D_x\Delta_y(\Gamma) = 0$, so the formula holds.

2. Γ has both hairs on the same vertex that is 2-valent. Because Γ is connected it must be $\Gamma = \sigma_2 =$, and the formula is easily checked.

3. Γ has both hairs on the same vertex y that is 3-valent. Let x be another end of the only edge at y, i.e. there is a flower on y with the root x. Then

$$\delta D^{(2)}(\Gamma) + D^{(2)}\delta(\Gamma) =$$
$$\sum_{\substack{w\\ w\neq x,y}}\left(\frac{1}{2}s_w D^{(2)}(\Gamma) - a_w D^{(2)}(\Gamma)\right) + D^{(2)}\sum_{\substack{w\\ w\neq x,y}}\left(\frac{1}{2}s_w(\Gamma) - a_w(\Gamma)\right) + D^{(2)}\left(\frac{1}{2}s_x(\Gamma) - a_x(\Gamma)\right) + D^{(2)}f_y(\Gamma)$$

where s'_y is the term of s_y that splits y to y and z where both hairs stay at y and the edge goes to z. Other terms of s_y cancel out with a_y and extracting the hair. Using similar arguments as in the proof of Proposition 5.12 one easily checks that

$$s_w D^{(2)}(\Gamma) = -D^{(2)}s_w(\Gamma),$$

$$\frac{1}{2}D^{(2)}s_x(\Gamma) = \sum_{\substack{w\\ w\neq x,y}} a_w D^{(2)}(\Gamma),$$

$$D^{(2)}a_x(\Gamma) = -\sum_{\substack{w\\ w\neq x,y}} D^{(2)}a_w(\Gamma),$$

$$2\nabla D^{(2)}s'_y(\Gamma) = D^{(1)}\Delta(\Gamma),$$

so the formula follows.

4. Γ has both hairs on the same vertex x that is more than 3-valent. Then by the definition $\delta D^{(2)}(\Gamma) = \delta(0) = 0$ and the only term that remains from $D^{(2)}\delta(\Gamma)$ is $D^{(2)}s'_x(\Gamma)$, where s'_x is the term of s_x that splits x to x and y where both hairs stay at x and all edges go to y. It still holds $2\nabla D^{(2)}s'_y(\Gamma) = D^{(1)}\Delta(\Gamma)$, so the formula follows

The proof of the second claim of the proposition is the same as the first case above. □

Note that the previous proposition fails for the disconnected graph $\Gamma\cup\sigma_2 \in \mathrm{H}_2\mathrm{fHGC}^{\circlearrowleft}_{-1,0}$ where $\Gamma \in \mathrm{H}_0\mathrm{fHGC}^{\circlearrowleft}_{-1,0}$.

Proposition 8.11. *For the connected part* $\mathrm{H}_3\mathrm{fHGCc}^{\circlearrowleft}_{-1,0}$ *it holds that*

$$\nabla D^{(2)}\Delta = 0.$$

Proof. The only possibility that $D^{(2)}\Delta(\Gamma)$ is not 0 is when $\Delta(\Gamma)$ has a flower, say at a vertex y with the root x. If an edge from x to y is created by Δ, y was disconnected from the rest of the graph in Γ, what is not possible. So the flower already existed in Γ. We have 2 cases:

1. The third hair in Γ is on the vertex $w \neq x$. Then the part of Δ that saves the flower deletes the hair on w and makes an edge between w and another vertex z that is not x. If $z = x$, $D^{(2)}$ will move it again to another vertex, cancelling the term where Δ sent it to its final destination immediately.

2. The third hair in Γ is on the vertex x. Δ deletes it and makes an edge between x and another vertex. $D^{(2)}$ then reconnects the edge from x to another vertex, so the resulting action is adding an edge in all possible ways. Then ∇ adds another edge in all possible ways. Adding one edge on one place and another edge on another place cancels with adding edges in the opposite order.

□

Note that the previous proposition fails for the disconnected graph $\Gamma\cup\sigma_3 \in \mathrm{H}_3\mathrm{fHGC}^{\circlearrowleft}_{-1,0}$ where $\Gamma \in \mathrm{H}_0\mathrm{fHGC}^{\circlearrowleft}_{-1,0}$.

8.3 Bounded hairy graph complexes

The general complex we will consider while proving Theorem 8.3 is $\left(\mathrm{fHGC}^{\geq 1\,\circlearrowleft}_{-1,n}, \delta + \Delta\right)$. In it the number of hairs h is unbounded. That makes the spectral sequences on the number of hairs h (that has δ as the first differential) and the one on the number of connected components c unbounded from above and we can not use standard results for convergence.

The complex splits as:

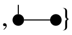
$$\left(\mathrm{fHGC}^{\geq 1\,\circlearrowleft}_{-1,n}, \delta + \Delta\right) = \prod_{f\in\mathbb{Z}} \left(\mathrm{F}_f\mathrm{fHGC}^{\geq 1\,\circlearrowleft}_{-1,n}, \delta + \Delta\right). \tag{8.3}$$

It is here $f = e + h - v$. In every $\mathrm{F}_f\mathrm{fHGC}^{\geq 1\,\circlearrowleft}_{-1,n}$ for the fixed degree $d = 1 + vn + (1-n)e - nh$, the number of edges $e = d - 1 + nf$ is fixed too. So increasing the number of hairs h increases the number of vertices v by the same amount. Since e is fixed, that increase will eventually force a graph to have connected components that are stars $\sigma_1 = \downarrow\!\bullet$. The purpose of this section is to show that components σ_1 are mostly irrelevant for the cohomology, so we can disallow them and bound h from above while preserving the result about cohomology.

We will also work with the similar complex $\mathrm{fHGC}^{\dagger\,\circlearrowleft}_{-1,n}$. Recall that $\mathrm{fHGC}^{\geq 1\,\circlearrowleft}_{-1,n}$ is the sub-complex of $\mathrm{fHGC}^{\circlearrowleft}_{-1,n}$ spanned by graphs that do not have $\sigma = \bullet$ as a connected component, while $\mathrm{fHGC}^{\dagger\,\circlearrowleft}_{-1,n}$ is the sub-complex of $\mathrm{fHGC}^{\circlearrowleft}_{-1,n}$ spanned by graphs that do not have σ or $\lambda = \bullet\!\!-\!\!\bullet$ as a connected component.

Definition 8.12. The *bounded hairy graph complex* $\mathrm{bHGC}^{\oslash}_{-1,n}$ is the subcomplex of $\mathrm{fHGC}^{\geq 1\,\oslash}_{-1,n}$ spanned by graphs that do not have an element of

$$H := \{\sigma_1, \lambda_2\} = \{\ ,\ \} \tag{8.4}$$



as a connected component.

The *unbounded hairy graph complex*, $\mathrm{uHGC}^{\oslash}_{-1,n}$ is the subcomplex of $\mathrm{fHGC}^{\geq 1\,\oslash}_{-1,n}$ spanned by graphs that do have at least one connected component in H, and also a connected component that is not in H.

The *unbounded remainder* $\mathrm{UR}_{-1,n}$ is the subcomplex of $\mathrm{fHGC}^{\geq 1\,\oslash}_{-1,n}$ spanned by graphs with all connected components in H.

Note that $\mathrm{uHGC}^{\oslash}_{-1,n}$ is not closed under the differentials Δ and $\delta + \Delta$. By abuse of notation, when we talk about the complex $\left(\mathrm{uHGC}^{\geq 1\,\oslash}_{-1,n}, \delta + \Delta\right)$ we actually mean the quotient $\left(\mathrm{fHGC}^{\dagger\,\oslash}_{-1,n}, \delta + \Delta\right) / \left(\mathrm{bHGC}^{\oslash}_{-1,n}, \delta + \Delta\right) \oplus (\mathrm{UR}_{-1,n}, \delta + \Delta)$, i.e. graphs that do not have a connected component in H or if all of them are in H are identified with zero.

Proposition 8.13.

$$H\left(\mathrm{fHGC}^{\ddagger\,\oslash}_{-1,n}, \delta\right) = H\left(\mathrm{fHGC}^{\geq 2\,\oslash}_{-1,n}, \delta\right) = H\left(\mathrm{bHGC}^{\dagger\,\oslash}_{-1,n}, \delta\right) = H\left(\mathrm{fHGC}^{\dagger\,\oslash}_{-1,n}, \delta\right) = H\left(\mathrm{fHGC}^{\oslash}_{-1,n}, \delta\right), \tag{8.5}$$

$$\begin{aligned} H\left(\mathrm{HGC}^{\oslash}_{m,n}, \delta\right) = H\left(\mathrm{HGC}_{m,n}, \delta\right) = H\left(\mathrm{H}_{\geq 1}\mathrm{bHGCc}^{\dagger\,\oslash}_{m,n}, \delta\right) = \\ = H\left(\mathrm{H}_{\geq 1}\mathrm{bHGCc}^{\dagger}_{m,n}, \delta\right) = H\left(\mathrm{H}_{\geq 1}\mathrm{bHGCc}^{\oslash}_{m,n}, \delta\right) = H\left(\mathrm{H}_{\geq 1}\mathrm{bHGCc}_{m,n}, \delta\right). \end{aligned} \tag{8.6}$$

Proof. Straightforward extension of Corollaries 2.26 and 2.27. □

Lemma 8.14. *$H\left(\mathrm{UR}_{-1,n}, \delta + \Delta\right)$ is 1-dimensional, the class being represented by*

$$\alpha = \sum_{n\geq 1} \frac{1}{n!} \sigma_1^{\cup n}. \tag{8.7}$$

Proof. On $\mathrm{UR}_{-1,n}$ we set up a spectral sequence on the number of vertices. The spectral sequence clearly converges correctly. The first differential is Δ. In the first row there is just σ_1, and it survives on the first page. All other rows have two terms, one with and one without λ_2, that cancel on the first page. Therefore, the cohomology must be 1-dimensional. One easily checks that $(\delta + \Delta)\alpha = 0$ and since α has σ_1 in the first row, it represents the class. □

Proposition 8.15. *The complex $\left(\mathrm{uHGC}^{\dagger\,\oslash}_{-1,n}, \delta + \Delta\right)$ is acyclic.*

Proof. Let $\tilde{\Gamma} \in \mathrm{uHGC}^{\dagger\,\oslash}_{-1,n}$. We write $\tilde{\Gamma} = \Gamma \cup \gamma$ where $\Gamma \in \mathrm{bHGC}^{\dagger\,\oslash}_{-1,n}$ and $\gamma \in \mathrm{UR}_{-1,n}$ and call Γ the *main part* of $\tilde{\Gamma}$ and γ the *secondary* part of $\tilde{\Gamma}$.

Let us set up a spectral sequence of $\left(\mathrm{uHGC}^{\dagger\,\oslash}_{-1,n}, \delta + \Delta\right)$ on the number of edges in the main part. It is easily seen that the differential can not decrease that number. To ensure the correct convergence we split unbounded complex similarly as the full complex:

$$\left(\mathrm{uHGC}^{\dagger\,\oslash}_{-1,n}, \delta + \Delta\right) = \prod_{f\in\mathbb{Z}} \left(\mathrm{F}_f\mathrm{uHGC}^{\dagger\,\oslash}_{-1,n}, \delta + \Delta\right).$$

In every $\mathrm{F}_f\mathrm{uHGC}^{\oslash}_{-1,n}$ for the fixed degree $d = 1 + vn + (1-n)e - nh$, the total number of edges $e = d - 1 + nf$ is fixed too, so the number of edges in the main part is bounded and Corollary B.3 implies correct convergence.

One can check that the first part of the differential is the one that acts within the secondary part only. Therefore the complexes on the first page are the direct product of complexes for the fixed main part, which are all clearly isomorphic to $\mathrm{UR}_{-1,n}$. Therefore, using Lemma 8.14, on the first page of the spectral sequence we have classes represented by $\Gamma \cup \alpha$ for $\Gamma \in \mathrm{bHGC}^{\dagger\,\oslash}_{-1,n}$.

On the second page matters the part of the differential that acts within the main part only, and the one that connects the main part to the secondary part. The element is now uniquely determined by the main part Γ so we can investigate what does the differential to it:

$$\Gamma \mapsto \delta(\Gamma) + \Delta(\Gamma) + \sum_{k \in H(\Gamma)} e_k + \sum_{x \in V(\Gamma)} a_x = \Delta(\gamma) + \sum_{x \in V(\Gamma)} \frac{1}{2} s_x .$$

The first sum corresponds to connecting a hair to one σ_1 in α, and the second sum corresponds to connecting a hair from one σ_1 to the main part.

The following lemma concludes the proof. $\square$

Lemma 8.16. *The complex* $\mathrm{bHGC}^{\dagger\varnothing}_{-1,n}$ *with the differential* $\Gamma \mapsto \Delta(\Gamma) + \sum_{x \in V(\Gamma)} \frac{1}{2} s_x$ *is acyclic.*

Proof. Let an *antenna* be a maximal connected sub-graph consisting of at least one 1-valent vertex and 2-valent vertices. Note that there are two kinds of linear graphs (those that do not have 3- or more-valent vertex) that are entirely considered an antenna: hairless one (that is longer than two vertices) and the one with a hair on one end. They can also be a connected component in a graph, and we call them *linear antennas*.

The length of an antenna is the number of vertices in it. Let l be the total length of all antennas in the graph. We set up a spectral sequence on the number $l - e$ (e is the number of edges). The differential can not increase that number. To ensure the correct converging of the spectral sequence, we split the complex into the product of complexes for fixed $c = e + h - v$ and use Corollary B.3. Here, the absence of a connected components σ_1 is needed.

It is easy to check that the first part of the differential is only extending an antenna. When summed all together, even-length antennas (including “0-length antennas”, i.e. vertices that are not in an antenna) are extended by one vertex, and odd-length antennas are sent to 0. For linear antennas it is the opposite. There is a homotopy that contracts an odd-length antenna and even-length linear antenna. There is always an antenna (at least 0-length) in the graph, so the homotopy implies the acyclicity. $\square$

Corollary 8.17. $H\left(\mathrm{uHGC}^{\varnothing}_{-1,0}, \delta + \Delta\right)$ *is one-dimensional, the class being represented by* $\lambda \cup \alpha$.

Proof. A graph in $\mathrm{uHGC}^{\varnothing}_{-1,0}$ can have at most one connected component λ because of the symmetry reasons. So it is of the form $\Gamma \in \mathrm{uHGC}^{\dagger\varnothing}_{-1,0}$ or $\Gamma \cup \lambda$ for $\Gamma \in \mathrm{uHGC}^{\dagger\varnothing}_{-1,0}$ or $\Gamma \in \mathrm{UR}_{-1,0}$. So

$$\left(\mathrm{uHGC}^{\varnothing}_{-1,0}, \delta + \Delta\right) = \left(\mathrm{uHGC}^{\dagger\varnothing}_{-1,0} \oplus \mathrm{uHGC}^{\dagger\varnothing}_{-1,0} \cup \lambda, \delta + \Delta\right) \oplus \left(\mathrm{UR}_{-1,0} \cup \lambda, \delta + \Delta\right).$$

On the first term we set up a spectral sequence on the number of λ-s (two rows) that clearly converges correctly, and on the first page we have two copies of $\left(\mathrm{uHGC}^{\dagger\varnothing}_{-1,0}, \delta + \Delta\right)$ that are acyclic by Proposition 8.15, so the whole complex is acyclic. The second term and Lemma 8.14 give the class. $\square$

Chapter 9

Extra differential for hairy graph complex, *m* odd, *n* odd

In this chapter we prove the first part of Theorem 8.3, i.e. that $H\left(\mathrm{HGC}_{-1,1}, \delta+\Delta\right)=0$. The following diagram describes the way to do that. 'Almost acyclic' means that there are only a few classes of cohomology that are easy to calculate.

$$
\begin{gathered}
\Big\downarrow 9.1\\
\left(\mathrm{fHGC}_{-1,1}^{\geq 1} \oplus \mathrm{fGC}_1^{\geq 1}[-3], \Delta+D^{(1)}\right) \text{ is almost acyclic, except for the classes without hairs}\\
\Big\downarrow 9.2\\
\left(\mathrm{fHGC}_{-1,1}^{\dagger} \oplus \mathrm{fGC}_1^{\dagger}[-3], \Delta+D^{(1)}\right) \text{ is almost acyclic, except for the classes without hairs}\\
\Big\downarrow 9.3\\
\left(\mathrm{H}_\flat \mathrm{fHGC}_{-1,1}^{\dagger}, \Delta+D^{(1)}\right) \text{ is almost acyclic}\\
\Big\downarrow 9.4\\
\left(\mathrm{H}_\flat \mathrm{fHGC}_{-1,1}^{\dagger}, \delta'+\Delta+D^{(1)}\right) \text{ is almost acyclic}\\
\Big\downarrow 9.5\\
\left(\mathrm{H}_\flat \mathrm{bHGC}_{-1,1}^{\dagger}, \delta'+\Delta+D^{(1)}\right) \text{ is acyclic}\\
\Big\downarrow 9.6\\
\left(\mathrm{H}_\flat \mathrm{fHGC}_{-1,1}^{\ddagger}, \delta'+\Delta+D^{(1)}\right) \text{ is acyclic}\\
\Big\downarrow 9.7\\
\left(\mathrm{H}_{\geq 1} \mathrm{fHGC}_{-1,1}^{\ddagger}, \delta+\Delta\right) \text{ is acyclic}\\
\Big\downarrow 9.8\\
\left(\mathrm{H}_{\geq 1} \mathrm{fHGCc}_{-1,1}^{\ddagger}, \delta+\Delta\right) \text{ is acyclic}\\
\Big\downarrow 9.9\\
\left(\mathrm{HGC}_{-1,1}, \delta+\Delta\right) \text{ is acyclic}
\end{gathered}
$$

Can this be shown in a shorter way? The easiest way to show that a complex with a differential $\delta+\Delta$ is acyclic is to make the spectral sequence in which the first differential is Δ, and to use the fact that the complex with the differential Δ is acyclic. But neither $\left(\mathrm{H}_{\geq 1}\mathrm{fHGC}_{-1,1}^{\geq 1}, \Delta\right)$ nor $\left(\mathrm{fHGC}_{-1,1}^{\geq 1}, \Delta\right)$ is acyclic, there are classes with 1 or no hairs. For technical reasons we need to disallow λ as a connected component and change the constraint ≥ 1 to $\dagger$. We then change the hairless part in $\left(\mathrm{fHGC}_{-1,1}^{\dagger}, \Delta\right)$ to kill classes with 1 or no hairs and make the new complex,

$\left(\mathrm{H}_\flat \mathrm{fHGC}^{\dagger}_{-1,1}, \Delta + D^{(1)}\right)$, almost acyclic (Corollary 9.3).

The spectral sequence argument now leads to the conclusion that the complex with the differential $\delta + \Delta$ is almost acyclic (Proposition 9.4). But it is not our intended result, there is a complicated hairless part. To remove it (Proposition 9.6), we need a change of constraint to ‡ (at least 2-valent vertices, at least 3-valent hairy vertices). The standard result that the change of constraint does not change the cohomology of the standard differential δ (Proposition 8.13), can be used in the spectral sequence with the standard differential being the first one (Proposition 9.6). But the spectral sequence is bounded, and hence converges correctly, only if we change to the bounded complex before that (Proposition 9.5).

Recall that for $m = -1$ and $n = 1$ the degree is $d = v + 1 - h$.

9.1 The differential Δ

In this section we want to study the cohomology of $\left(\mathrm{fHGC}^{\geq 1}_{-1,1}, \Delta\right)$. We will actually study a slightly different complex with an extra term $\mathrm{H}_0\mathrm{fHGC}^{\geq 1}_{-1,1}[-1]$:

$$\left(\mathrm{fHGC}^{\geq 1}_{-1,1} \oplus \mathrm{H}_0\mathrm{fHGC}^{\geq 1}_{-1,1}[-1], \Delta + D^{(1)}\right) \tag{9.1}$$

where $\Delta : \mathrm{fHGC}^{\geq 1}_{-1,1} \to \mathrm{fHGC}^{\geq 1}_{-1,1}$ and $D^{(1)} : \mathrm{H}_1\mathrm{fHGC}^{\geq 1}_{-1,1} \to \mathrm{H}_0\mathrm{fHGC}^{\geq 1}_{1}[-1]$ is the operator of deleting a hairy vertex defined in Definition 8.4. Recall that the degree is $d = v + 1 - h$, so the degree shift $[-1]$ is necessarily to make the differential of the degree $+1$. Proposition 8.10 shows that $D^{(1)}\Delta = 0$ in $\mathrm{fHGC}^{\geq 1}_{-1,1}$, so the differential indeed squares to 0.

Recall (2.12), so $\mathrm{H}_0\mathrm{fHGC}^{\geq 1}_{-1,1}[-1] = \mathrm{fGC}^{\geq 1}_{1}[-3]$. For simplicity we use the latter notation.

Our new complex splits as the product of subcomplexes with fixed number of vertices v, with the extra term having $v - 1$ vertices:

$$\left(\mathrm{fHGC}^{\geq 1}_{-1,1} \oplus \mathrm{fGC}^{\geq 1}_{1}[-3], \Delta + D^{(1)}\right) = \prod_{v \geq 1} \left(\mathrm{V}_v\mathrm{fHGC}^{\geq 1}_{-1,1} \oplus \mathrm{V}_{v-1}\mathrm{fGC}^{\geq 1}_{1}[-3], \Delta + D^{(1)}\right) \tag{9.2}$$

In each subcomplex, the degree $d = v + 1 - h$ is up to the shift equal to the negative number of hairs $-h$. We may write it as:

$$\begin{tikzcd}
d = & v-2 & v-1 & v & v+1 \\
\cdots \arrow[r, "\Delta"] & \mathrm{H}_3\mathrm{V}_v\mathrm{fHGC}^{\geq 1}_{-1,1} \arrow[r, "\Delta"] & \mathrm{H}_2\mathrm{V}_v\mathrm{fHGC}^{\geq 1}_{-1,1} \arrow[r, "\Delta"] & \mathrm{H}_1\mathrm{V}_v\mathrm{fHGC}^{\geq 1}_{-1,1} \arrow[r, "\Delta"] \arrow[dr, "D^{(1)}"'] & \mathrm{H}_0\mathrm{V}_v\mathrm{fHGC}^{\geq 1}_{-1,1} \\
& & & & \oplus \\
& & & & \mathrm{V}_{v-1}\mathrm{fGC}^{\geq 1}_{1}[-3]
\end{tikzcd}$$

Proposition 9.1.

- $H\left(\mathrm{V}_1\mathrm{fHGC}^{\geq 1}_{-1,1}, \Delta\right)$ *is one-dimensional, the class being represented by* $\sigma_1 =$ *;*
- $H_d\left(\mathrm{V}_v\mathrm{fHGC}^{\geq 1}_{-1,1} \oplus \mathrm{V}_{v-1}\mathrm{fGC}^{\geq 1}_{1}[-3], \Delta + D^{(1)}\right) = 0$ *for* $v > 1$ *and* $d \leq v$.

Note that the second claim of the proposition does not say anything about the cohomology at degree $d = v + 1$, and for $d \geq v + 2$ it is trivially 0.

Proof. $\mathrm{V}_{v-1}\mathrm{fGC}^{\geq 1}_{1}[-3]$ is isomorphic to the subcomplex of $\mathrm{V}_v\mathrm{fGC}_1[-2]$ spanned by graphs with an isolated vertex, the isomorphism being adding an isolated vertex $\cup\sigma_0$. Since the proposition does not say anything about the cohomology at degree $v + 1$ we may safely replace $\mathrm{V}_{v-1}\mathrm{fGC}^{\geq 1}_{1}[-3]$ with the whole $\mathrm{V}_v\mathrm{fGC}_1[-2]$. The purpose is to make $D^{(1)}$ not change the number of vertices, so the new $D^{(1)} : \mathrm{V}_v\mathrm{H}_1\mathrm{fHGC}^{\geq 1}_{-1,1} \to \mathrm{V}_v\mathrm{fGC}_1[-2]$ reconnects all edges from the hairy vertex and deletes it, but restores the vertex without its hair.

Since the differential does not change the number of vertices, we can use Proposition A.1 and work with fixed number of vertices and distinguish them: let $V_v := \bar{\mathrm{V}}_v\mathrm{fHGC}^{\geq 1}_{-1,1}$ and $W_v := \bar{\mathrm{V}}_v\mathrm{fGC}_1[-2]$.

The whole complex $\mathrm{V}_1\mathrm{fHGC}^{\geq 1}_{-1,1}$ is one dimensional and generated by σ_1, so the first statement of the proposition is clear.

We will show that for $v > 1$ $H_{v+1-h}\left(V_v + W_v, \Delta + D^{(1)}\right) = 0$ for $h \geq 1$ by induction on v. That will conclude the proof of the proposition. The claim includes the claim that $H_{v+1-h}(V_v, \Delta) = 0$ for $h \geq 2$. It is also true for

$v = 1$, and it will actually be used as the assumption of the induction. So, we may use the case $v = 1$ as the base of the induction. Suppose now that the assumption is true for $v - 1$ vertices, $v \geq 2$.

On V_v we choose one vertex, say the last one, and set up a spectral sequence on the total valence s of non-chosen vertices. So, an edge between non-chosen vertices counts twice, a hair on non-chosen vertex and an edge between non-chosen vertex and the chosen vertex counts ones, and hair on the chosen vertex does not count. The differential Δ can not decrease s and splits $\Delta = \Delta_0 + \Delta_1$ where Δ_0 is the part that does not change s. Δ_0 connects a hair from a non-chosen vertex to the chosen vertex and Δ_1 connects something to a non-chosen vertex increasing s always by 1.

We can furthermore split the complex as the product of subcomplexes with fixed $a = e + h$:

$$(V_v, \Delta) = \prod_{a \in \mathbb{Z}} (\mathrm{A}_a V_v, \Delta)\,.$$

For fixed degree $v + 1 - h$, i.e. fixed number of hairs h, s can get only finitely possible values, so the spectral sequence converges correctly in each $(\mathrm{A}_a V_v, \Delta)$, and therefore in the whole (V_v, Δ) using Corollary B.3.

On the first page of the spectral sequence there is the cohomology (V_v, Δ_0). Let $\beta : V_v \to V_v$ be the sum over all edges at the chosen vertex of deleting that edge (as heading towards the chosen edge for the matter of sign) and putting a hair on non-chosen vertex that was connected to that edge, unless it makes the chosen vertex 0-valent, being forbidden by definition. If the chosen vertex is not hairless and 1-valent, it is clear that $\Delta_0\beta + \beta\Delta_0 = C\mathrm{Id}$ where C is the number of edges at the chosen vertex plus the number of hairs on non-chosen vertices. So $H(V_v, \Delta_0) = 0$ unless the chosen vertex is isolated with a hair and there are no other hairs, or it is hairless 1-valent vertex.

By gluing a 1-vertex graph $\cup\sigma_0$ or $\cup\sigma_1$ in complexes V_v with distinguishable vertices we mean adding a new vertex with the highest number, so that it becomes chosen.

Every graph of the form $\Gamma \cup \sigma_1$, $\Gamma \in \mathrm{H}_0 V_{v-1}$, clearly represents a cohomology class. Let us call it a class of the *first type*. Graphs with a hairless 1-valent chosen vertex would not form classes if the vertex would be allowed to be 0-valent. Therefore, cutting that possibility implies that classes are represented by graphs of the form $\Delta_0(\Gamma \cup \sigma_0) =: c(\Gamma)$ for $\Gamma \in V_{v-1}$. Let us call them classes of the *second type*. It is easily seen that c is an isomorphism of degree 2, so classes of the second type on the second page of the spectral sequence are indeed equal to the classes of $H(V_{v-1}[-2])$. Classes are sketched in Figure 9.1.

$$\begin{array}{ccccccccc}
 & \times & & & & & & & \\
 & & \times & & & & & & \\
 & & & \times & & & & & \\
 & & & & \times & & & & \\
 & & & & & \times & & & \\
 & & & & & & \otimes & & \\
 & & & & & & & \times & \\
\cdots & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0
\end{array}$$

Figure 9.1: Classes on the first page of the spectral sequence of V_v. The numbers at the bottom are the numbers of hairs h, while the degree is $d = v + 1 - h$. Classes of the second type are labelled by $\times$ and the position where there are both classes is labelled by $\otimes$.

Let C be the complex isomorphic to the line of the second-type classes on the second page of the spectral sequence and it is depicted in Figure 9.2. $\mathrm{H}_h V_{v-1}$ is part of degree $d = v - h$ in V_{v-1}, and of degree $d = v + 2 - h$ in $V_{v-1}[-2]$. Here we set up the 2-row spectral sequence with the first type class $\Gamma \cup \sigma_1$, $\Gamma \in \mathrm{H}_0 V_{v-1}$, in the first row, and $V_{v-1}[-2]$ represented second type classes in the second row. First type class $\Gamma \cup \sigma_1$ is sent to the graph obtained from Γ by adding an antenna in all possible ways. The isomorphic element in $\mathrm{H}_1 V_{v-1}$ is $\chi^1(\Gamma)$, where χ^1 adds a hair in all possible ways.

The second row is, by the induction hypothesis, acyclic for the degree $d \leq v$. We do not know whether the degree $d = v$ first type class Γ from the first row is cancelled with something in the second row at the degree $d = v + 1$. If it is, $H_d(V_v) = 0$ for degree $d \leq v$.

If it is not, $H_d(V_v) = 0$ for degree $d \leq v-1$ and there is a class in degree $d = v$ in C represented by $\Gamma + \Gamma_2(\Gamma)$ for $\Gamma_2(\Gamma) \in \mathrm{H}_2 V_{v-1}$ (Γ_2 depends on Γ), where $\chi^1(\Gamma) + \Delta(\Gamma_2(\Gamma)) = 0$. Back in the starting complex V_v the isomorphic element is $\Gamma \cup \sigma_1 + \Delta_0(\Gamma_2(\Gamma) \cup \sigma_0)$ from the second page. It is clearly sent to 0 by the whole Δ, so it represents a class of the starting complex V_v, in the degree $d = v$.

$$\begin{tikzcd}[column sep=small]
 & & & & \mathrm{H}_0 V_{v-1} \arrow[dr, "\chi^1"] & \\
\cdots \arrow[r, "\Delta"'] & \mathrm{H}_5 V_{v-1} \arrow[r, "\Delta"'] & \mathrm{H}_4 V_{v-1} \arrow[r, "\Delta"'] & \mathrm{H}_3 V_{v-1} \arrow[r, "\Delta"'] & \mathrm{H}_2 V_{v-1} \arrow[r, "\Delta"'] & \mathrm{H}_1 V_{v-1} \\
d = & v-3 & v-2 & v-1 & v & v+1
\end{tikzcd}$$

Figure 9.2: The complex C split into the another 2-row spectral sequence.

Now we come back to the whole complex $\left(V_v + W_v, \Delta + D^{(1)}\right)$. Let us set up a spectral sequence of three rows: V_v, the part of W_v where the chosen vertex is isolated, and the rest of W_v (see Figure 9.3). The spectral sequence clearly converges correctly. On the first page in the first row in degree $d = v$ ($h = 1$) we may have classes represented by $\Gamma \cup \sigma_1 + \Delta_0(\Gamma_2(\Gamma) \cup \sigma_0)$ for some $\Gamma \in \mathrm{H}_0 V_{v-1}$ and $\Gamma_2(\Gamma) \in \mathrm{H}_2 V_{v-1}$, as shown above. In the degree $d = v + 1$ ($h = 0$) of the first page we are not interested. In the other rows in the degree $d = v + 1$ ($h = 0$) there is still the whole space, particularly in the second row there is $\Gamma \cup \sigma_0$ for every $\Gamma \in \mathrm{H}_0 V_{v-1}$.

$$\begin{tikzcd}
\cdots \arrow[r, "\Delta"] & \mathrm{H}_3 V_v \arrow[r, "\Delta"] & \mathrm{H}_2 V_v \arrow[r, "\Delta"] & \mathrm{H}_1 V_v \arrow[r, "\Delta"] \arrow[dr, "D^{(1)}_{\text{chosen vertex}}"] \arrow[ddr, "D^{(1)}_{\text{other vertices}}"'] & \mathrm{H}_0 V_v \\
 & & & & \text{Part of } W_v \text{ with chosen vertex isolated} \\
 & & & & \text{The rest of } W_v
\end{tikzcd}$$

Figure 9.3: Complex $\left(V_v + W_v, \Delta + D^{(1)}\right)$ split into 3-row spectral sequence.

On the second page $\Gamma \cup \sigma_1 + \Delta_0(\Gamma_2(\Gamma) \cup \sigma_0)$ is mapped by part of $D^{(1)}$ to the part of W_v where the chosen vertex is isolated, i.e. chosen vertex has been deleted (and restored). The only part of $\Gamma \cup \sigma_1 + \Delta(\Gamma_2(\Gamma) \cup \sigma_0)$ that has a hair on the chosen vertex is $\Gamma \cup \sigma_1$, so the differential is actually $\Gamma \cup \sigma_1 + \Delta(\Gamma_2(\Gamma) \cup \sigma_0) \mapsto \Gamma \cup \sigma_0$. It is clearly an injection, making the cohomology in the degree $d = v$ acyclic.

So, in both cases, if the class represented by $\Gamma \cup \sigma_1 + \Delta_0(\Gamma_2(\Gamma) \cup \sigma_0)$ in degree $d = v$ of $H(V_v, \Delta)$ exists or not, in the cohomology of the whole complex $H_d\left(V_v + W_v, \Delta + D^{(1)}\right) = 0$ for degrees $d \leq v$. That was to be demonstrated. □

9.2 Removing λ

In this section we transform the result to the complex $\mathrm{fHGC}^{\dagger}_{-1,1}$ that does not have $\lambda = \bullet\!\!-\!\!\bullet$ as a connected component.

Proposition 9.2.

- $H\left(\mathrm{V}_1\mathrm{fHGC}^{\dagger}_{-1,1}, \Delta\right)$ *is one-dimensional, the class being represented by* $\sigma_1 =$ ⫯*;*
- $H_d\left(\mathrm{V}_v\mathrm{fHGC}^{\dagger}_{-1,1} \oplus \mathrm{V}_{v-1}\mathrm{fGC}^{\dagger}_1[-3], \Delta + D^{(1)}\right) = 0$ *for* $v > 1$ *and* $d \leq v$*.*

Proof. We again do the proof on induction on v. For $v = 1$ there can not be any λ, so $\mathrm{V}_1\mathrm{fHGC}^{\dagger}_{-1,1} = \mathrm{V}_1\mathrm{fHGC}^{\geq 1}_{-1,1}$, so the result is indeed the same as in the Proposition 9.1. For $v = 2$ there is λ only in the hairless part and it represents a cohomology class, so in degrees we are considering it does not change the result of Proposition 9.1 either.

Let us pick $v > 2$ and assume that the proposition holds for any number of vertices smaller than v.

On $\left(\mathrm{V}_v\mathrm{fHGC}^{\geq 1}_{-1,1} \oplus \mathrm{V}_{v-1}\mathrm{fGC}^{\geq 1}_1[-3], \Delta + D^{(1)}\right)$ we set up a spectral sequence on the number of λ-s. The differential can not increase that number, and it is bounded, so the spectral sequence converges correctly. The lowest row on the first page is our intended complex $\left(\mathrm{V}_v\mathrm{fHGC}^{\dagger}_{-1,1} \oplus \mathrm{V}_{v-1}\mathrm{fGC}^{\dagger}_1[-3], \Delta + D^{(1)}\right)$.

The first differential in the other rows does not effect any λ, so it is the same as the complex without them, but now with fewer vertices (by 2, 4, etc.), with a degree shift. All of them are acyclic by the assumption of induction in degrees that correspond to more than one hair ($d \leq v-1$). Therefore, if there is a class with a hair ($d \leq v$) in the last row, it can not be cancelled by anything, contradicting the result that the whole complex is acyclic in that degrees (Proposition 9.1). □

To simplify the result we define another complex

$$\mathrm{H}_\flat \mathrm{V}_v \mathrm{fHGC}^\dagger_{-1,1} := \mathrm{H}_{\geq 1}\mathrm{V}_v \mathrm{fHGC}^\dagger_{-1,1} \oplus \left(\Delta + D^{(1)}\right)\left(\mathrm{H}_1 \mathrm{V}_v \mathrm{fHGC}^\dagger_{-1,1}\right). \tag{9.3}$$

We have changed the term with the highest degree $\mathrm{H}_0\mathrm{V}_v\mathrm{fHGC}^\dagger_{-1,1} \oplus \mathrm{V}_{v-1}\mathrm{fGC}^\dagger_1[-3]$ with its subspace $\left(\Delta + D^{(1)}\right)\left(\mathrm{H}_1\mathrm{V}_v\mathrm{fHGC}^\dagger_{-1,1}\right)$, the image of the differential, to ensure the acyclicity at that degree. The whole complex, including all numbers of vertices, is

$$\mathrm{H}_\flat \mathrm{fHGC}^\dagger_{-1,1} := \prod_{v\in\mathbb{N}} \mathrm{H}_\flat \mathrm{V}_v \mathrm{fHGC}^\dagger_{-1,1} = \mathrm{H}_{\geq 1}\mathrm{fHGC}^\dagger_{-1,1} \oplus \mathrm{HL}_1, \tag{9.4}$$

where

$$\mathrm{HL}_1 := \left(\Delta + D^{(1)}\right)\left(\mathrm{H}_1 \mathrm{fHGC}^\dagger_{-1,1}\right) \subset \mathrm{H}_0\mathrm{fHGC}^\dagger_{-1,1} \oplus \mathrm{fGC}^\dagger_1[-3] \tag{9.5}$$

is the hairless part.

Corollary 9.3. *$H\left(\mathrm{H}_\flat\mathrm{fHGC}^\dagger_{-1,1}, \Delta + D^{(1)}\right)$ is one-dimensional, the class being represented by σ_1.*

9.3 The differential $\delta + \Delta$

On $\mathrm{fHGC}^\dagger_{-1,1}$ there is the standard differential δ. We extend it to $\delta' : \mathrm{fHGC}^\dagger_{-1,1} \oplus \mathrm{fGC}^{\geq 1}_1[-3] \to \mathrm{fHGC}^\dagger_{-1,1} \oplus \mathrm{fGC}^{\geq 1}_1[-3]$ as follows

$$\delta'(\Gamma, \gamma) = (\delta(\Gamma), \mathrm{H}_0\Gamma - \delta(\gamma)), \tag{9.6}$$

where $\mathrm{H}_h\Gamma$ is part of Γ with h hairs. It clearly squares to 0 and has degree 1, so it is a differential.

Differentials δ and Δ anti-commute, so $\left(\mathrm{H}_{\geq 1}\mathrm{fHGC}^\dagger_{-1,1}, \delta + \Delta\right)$ is a complex. Proposition 8.8 implies

$$\begin{aligned}\left(\Delta + D^{(1)}\right)\delta'(\Gamma,\gamma) + \delta'\left(\Delta + D^{(1)}\right)(\Gamma,\gamma) &= \left(\Delta + D^{(1)}\right)\left(\delta(\Gamma), H^0\Gamma - \delta(\gamma)\right) + \delta'\left(\Delta(\Gamma), D^{(1)}\mathrm{H}_1\Gamma\right) = \\ &= \left(\Delta\delta(\Gamma), D^{(1)}\mathrm{H}_1\delta(\Gamma)\right) + \left(\delta\Delta(\Gamma), \mathrm{H}_0\Delta(\Gamma) - \delta D^{(1)}\mathrm{H}_1\Gamma\right) = (0,0),\end{aligned}$$

i.e. δ' and $\Delta + D^{(1)}$ anti-commute and $\left(\mathrm{H}_{\geq 1}\mathrm{fHGC}^\dagger_{-1,1} \oplus \mathrm{fGC}^\dagger_1[-3], \delta' + \Delta + D^{(1)}\right)$ is also a complex. Because of the same reason the restriction $\delta' : \mathrm{H}_\flat\mathrm{fHGC}^\dagger_{-1,1} \to \mathrm{H}_\flat\mathrm{fHGC}^\dagger_{-1,1}$ is well defined, so $\left(\mathrm{H}_\flat\mathrm{fHGC}^\dagger_{-1,1}, \delta'\right)$ and $\left(\mathrm{H}_\flat\mathrm{fHGC}^\dagger_{-1,1}, \delta' + \Delta + D^{(1)}\right)$ are also complexes.

Proposition 9.4. *$H\left(\mathrm{H}_\flat\mathrm{fHGC}^\dagger_{-1,1}, \delta' + \Delta + D^{(1)}\right)$ is one-dimensional, the class being represented by $\alpha = \sum_{n\geq 1}\frac{1}{n!}\sigma_1^{\cup n}$.*

Proof. We set up a spectral sequence on v from the splitting (9.2), such that the first differential is $\Delta + D^{(1)}$. By Corollary 9.3 on the first page survives only σ_1.

We can split the complex $\left(\mathrm{fHGC}^\dagger_{-1,1} \oplus \mathrm{fGC}^\dagger_1[-3], \delta' + \Delta + D^{(1)}\right)$ as the product of subcomplexes with fixed $f = e + h - v$:

$$\left(\mathrm{fHGC}^\dagger_{-1,1} \oplus \mathrm{fGC}^\dagger_1[-3], \delta' + \Delta + D^{(1)}\right) = \prod_{f\in\mathbb{Z}} \left(\mathrm{F}_f\mathrm{fHGC}^\dagger_{-1,1} \oplus \mathrm{F}_f\mathrm{fGC}^\dagger_1[-3], \delta' + \Delta + D^{(1)}\right).$$

The same splitting can be done for the subcomplex $\mathrm{H}_\flat\mathrm{fHGC}^\dagger_{-1,1}$:

$$\left(\mathrm{H}_\flat\mathrm{fHGC}^\dagger_{-1,1}, \delta' + \Delta + D^{(1)}\right) = \prod_{f\in\mathbb{Z}} \left(\mathrm{F}_f\mathrm{H}_\flat\mathrm{fHGC}^\dagger_{-1,1}, \delta' + \Delta + D^{(1)}\right).$$

For fixed v and degree $d = v + 1 - h$ the numbers of edges e and hairs h are bounded in each $\mathrm{F}_f\mathrm{H}_\flat\mathrm{fHGC}^\dagger_{-1,1}$, so Corollary B.3 implies the spectral sequence converges correctly.

The cohomology of $\left(\mathrm{H}_\flat\mathrm{fHGC}^\dagger_{-1,1}, \delta' + \Delta + D^{(1)}\right)$ is therefore one-dimensional. One checks that α is mapped to 0, and since σ_1 is the highest part of α, α represents the class in $H\left(\mathrm{H}_\flat\mathrm{fHGC}^\dagger, \delta' + \Delta + D^{(1)}\right)$. □

9.4 Bounded complex

Let

$$\mathrm{H}_\flat\mathrm{bHGC}^{\dagger}_{-1,1} := \mathrm{H}_{\geq 1}\mathrm{bHGC}^{\dagger}_{-1,1} \oplus \mathrm{HL}_1 \subset \mathrm{H}_\flat\mathrm{fHGC}^{\dagger}_{-1,1}, \tag{9.7}$$

where HL_1 is as in (9.5) and $\mathrm{bHGC}_{-1,1}$ is defined in Definition 8.12.

Proposition 9.5. *The complex* $\left(\mathrm{H}_\flat\mathrm{bHGC}^{\dagger}_{-1,1}, \delta' + \Delta + D^{(1)}\right)$ *is acyclic.*

Proof. It holds that

$$\left(\mathrm{H}_\flat\mathrm{fHGC}^{\dagger}_{-1,1}, \delta' + \Delta + D^{(1)}\right) = (\mathrm{UR}_{-1,1}, \delta + \Delta) \oplus \left(\mathrm{uHGC}_{-1,1} \oplus \mathrm{H}_\flat\mathrm{bHGC}^{\dagger}_{-1,1}, \delta' + \Delta + D^{(1)}\right).$$

Proposition 9.4 says that $H\left(\mathrm{H}_\flat\mathrm{fHGC}^{\dagger}_{-1,1}, \delta' + \Delta + D^{(1)}\right)$ has one class $[\alpha]$. It can easily be seen that this class belongs to $(\mathrm{UR}_{-1,1}, \delta + \Delta)$, so the complex $\left(\mathrm{uHGC}_{-1,1} \oplus \mathrm{H}_\flat\mathrm{bHGC}^{\dagger}_{-1,1}, \delta' + \Delta + D^{(1)}\right)$ is acyclic.

On $\left(\mathrm{uHGC}_{-1,1} \oplus \mathrm{H}_\flat\mathrm{bHGC}^{\dagger}_{-1,1}, \delta' + \Delta + D^{(1)}\right)$ we set up a spectral sequence of two obvious rows: $\mathrm{uHGC}_{-1,1}$ and $\mathrm{H}_\flat\mathrm{bHGC}^{\dagger}_{-1,1}$. The spectral sequence clearly converges correctly. Proposition 8.15 implies that the first row is acyclic, so the second row has to be acyclic too. That was to be demonstrated. □

9.5 At least 2-valent vertices

Let

$$\mathrm{H}_\flat\mathrm{fHGC}^{\ddagger}_{-1,1} := \mathrm{H}_{\geq 1}\mathrm{fHGC}^{\ddagger}_{-1,1} \oplus \mathrm{HL}_1 \subset \mathrm{H}_\flat\mathrm{bHGC}^{\dagger}_{-1,1}. \tag{9.8}$$

Recall that $\mathrm{fHGC}^{\ddagger}_{-1,1}$ is the complex spanned by graphs whose vertices are at least 2-valent, and hairy vertices are at least 3-valent.

Proposition 9.6. *The inclusion* $\left(\mathrm{H}_\flat\mathrm{fHGC}^{\ddagger}_{-1,1}, \delta' + \Delta + D^{(1)}\right) \hookrightarrow \left(\mathrm{H}_\flat\mathrm{bHGC}^{\dagger}_{-1,1}, \delta' + \Delta + D^{(1)}\right)$ *is a quasi- -isomorphism.*

Proof. We will show that the mapping cone is acyclic. On it let us set up a spectral sequence on the number of hairs h.

We again split complexes as the product of subcomplexes with fixed $f = e + h - v$:

$$\left(\mathrm{H}_\flat\mathrm{fHGC}^{\ddagger}_{-1,1}, \delta' + \Delta + D^{(1)}\right) = \prod_{f\in\mathbb{Z}} \left(\mathrm{F}_f\mathrm{H}_\flat\mathrm{fHGC}^{\ddagger}_{-1,1}, \delta' + \Delta + D^{(1)}\right),$$

$$\left(\mathrm{H}_\flat\mathrm{bHGC}^{\dagger}_{-1,1}, \delta' + \Delta + D^{(1)}\right) = \prod_{f\in\mathbb{Z}} \left(\mathrm{F}_f\mathrm{H}_\flat\mathrm{bHGC}^{\dagger}_{-1,1}, \delta' + \Delta + D^{(1)}\right).$$

For fixed degree $d = v + 1 - h$ the number of edges e is fixed in each $\mathrm{F}_f\mathrm{H}_\flat\mathrm{fHGC}^{\ddagger}_{-1,1}$ and $\mathrm{F}_f\mathrm{H}_\flat\mathrm{bHGC}^{\dagger}_{-1,1}$. Also, increasing the number of hairs h increases the number of vertices v by the same amount. Since the number of edges e is fixed, for h big enough, there will be an isolated vertex. But that is not possible in either $\mathrm{H}_\flat\mathrm{fHGC}^{\ddagger}_{-1,1}$ or $\mathrm{H}_\flat\mathrm{bHGC}^{\dagger}_{-1,1}$. So, the spectral sequence of the mapping cone of the inclusion $\left(\mathrm{F}_f\mathrm{H}_\flat\mathrm{fHGC}^{\ddagger}_{-1,1}, \delta' + \Delta + D^{(1)}\right) \hookrightarrow \left(\mathrm{F}_f\mathrm{H}_\flat\mathrm{bHGC}^{\dagger}_{-1,1}, \delta' + \Delta + D^{(1)}\right)$ is bounded above and converges correctly for every f, and therefore also the spectral sequence of the mapping cone of the whole inclusion $\left(\mathrm{H}_\flat\mathrm{fHGC}^{\ddagger}_{-1,1}, \delta' + \Delta + D^{(1)}\right) \hookrightarrow \left(\mathrm{H}_\flat\mathrm{bHGC}^{\dagger}_{-1,1}, \delta' + \Delta + D^{(1)}\right)$ converges correctly by Corollary B.3. This convergence is the very reason why we introduced the bounded complex.

On the first page of the spectral sequence, for $h = 0$ there is a mapping cone of the identity $(\mathrm{HL}_1, \delta') \to (\mathrm{HL}_1, \delta')$, so it is acyclic. For $h > 0$ there is a mapping cone of the inclusion $\left(\mathrm{fHGC}^{\ddagger}_{-1,1}, \delta\right) \to \left(\mathrm{bHGC}^{\dagger}_{-1,1}, \delta\right)$. It is acyclic by Proposition 8.13. That concludes the proof. □

9.6 Removing the hairless part

Proposition 9.7. *The complex* $\left(\mathrm{H}_{\geq 1}\mathrm{fHGC}^{\ddagger}_{-1,1}, \delta + \Delta\right)$ *is acyclic.*

Proof. On $\left(\mathrm{H}_{\flat}\mathrm{fHGC}^{\ddagger}_{-1,1}, \delta' + \Delta + D^{(1)}\right)$ we set up a spectral sequence of two obvious rows: $\mathrm{H}_{\geq 1}\mathrm{fHGC}^{\ddagger}_{-1,1}$ and HL_1. Clearly the spectral sequence converges correctly. Propositions 9.5 and 9.6 imply that the whole complex is acyclic, so all classes of the first page cancel out. We claim that there are no classes on the first page.

Suppose the opposite, that there is a class in the first row represented by $\Gamma \in \mathrm{H}_{\geq 1}\mathrm{fHGC}^{\ddagger}_{-1,1}$. Let $\Delta = \Delta_0 + \Delta_1$ where $\Delta_0 : \mathrm{H}_{\geq 1}\mathrm{fHGC}^{\ddagger}_{-1,1} \to \mathrm{H}_{\geq 1}\mathrm{fHGC}^{\ddagger}_{-1,1}$ and $\Delta_1 : \mathrm{H}_{\geq 1}\mathrm{fHGC}^{\ddagger}_{-1,1} \to \mathrm{HL}_1$. It holds that $(\delta + \Delta_0)(\Gamma) = 0$ and $\left(\Delta_1 + D^{(1)}\right)(\Gamma)$ represents a class in HL_1. We write $\Gamma = \sum_{h\geq 1} \mathrm{H}_h\Gamma$. Then it holds that

$$\Delta\left(\mathrm{H}_2\Gamma\right) + \delta\left(\mathrm{H}_1\Gamma\right) = 0$$

and $\left(\Delta + D^{(1)}\right)(\mathrm{H}_1\Gamma)$ represents a class in HL_1.

Let $\gamma := D^{(p)}\left(\mathrm{H}_1\Gamma\right)$. It is for sure in $\mathrm{H}_1\mathrm{fHGC}^{\dagger}_{-1,1}$. Lemma 8.6 implies

$$\Delta(\gamma) = D^{(1)}\left(\mathrm{H}_1\Gamma\right),$$

and Lemma 8.7 implies

$$D^{(1)}(\gamma) = 0.$$

This relation was the very reason of introducing the constraint ‡. Propositions 8.8 and 8.10 imply that

$$\Delta\left(\mathrm{H}_1\Gamma\right) = \delta D^{(1)}\left(\mathrm{H}_1\Gamma\right) - D^{(1)}\delta\left(\mathrm{H}_1\Gamma\right) = \delta D^{(1)}\left(\mathrm{H}_1\Gamma\right) + D^{(1)}\Delta\left(\mathrm{H}_2\Gamma\right) = \delta D^{(1)}\left(\mathrm{H}_1\Gamma\right).$$

The equalities are diagrammatically expressed in Figure 9.4.

$$\begin{tikzcd}
 & \mathrm{H}_2\Gamma \arrow[dr, "-\Delta"] & \\
\gamma \arrow[dr, "\Delta"] \arrow[ddr, "D^{(1)}"'] & \mathrm{H}_1\Gamma \arrow[l, "D^{(p)}"'] \arrow[r, "\delta"] \arrow[d, "D^{(1)}"'] \arrow[dr, "\Delta"] \arrow[ddr] & \delta\left(\mathrm{H}_1\Gamma\right) \\
 & D^{(1)}\left(\mathrm{H}_1\Gamma\right) \arrow[r, "\delta"] \arrow[dr, "\mathrm{Id}"'] & \Delta\left(\mathrm{H}_1\Gamma\right) \\
 & 0 & D^{(1)}\left(\mathrm{H}_1\Gamma\right)
\end{tikzcd}$$

Figure 9.4: Maps in $\mathrm{H}_{\geq 1}\mathrm{fHGC}^{\ddagger}_{-1,1} \oplus \mathrm{H}_0\mathrm{fHGC}^{\dagger}_{-1,1} \oplus \mathrm{fGC}^{\dagger}_1[-3]$. The rows are, from the bottom up: $\mathrm{fGC}^{\dagger}_1[-3]$, $\mathrm{H}_0\mathrm{fHGC}^{\dagger}_{-1,1}$, $\mathrm{H}_1\mathrm{fHGC}^{\ddagger}_{-1,1}$ and $\mathrm{H}_2\mathrm{fHGC}^{\ddagger}_{-1,1}$.

Note that the complex in the figure is bigger than $\mathrm{H}_{\flat}\mathrm{fHGC}^{\ddagger}_{-1,1}$, its hairless part is the whole $\mathrm{H}_0\mathrm{fHGC}^{\dagger}_{-1,1} \oplus \mathrm{fGC}^{\dagger}_1[-3]$ instead of only HL_1. Indeed, both mentioned elements of the hairless part $\left(D^{(1)}\left(\mathrm{H}_1\Gamma\right), 0\right)$ and $\left(\delta\left(\mathrm{H}_1\Gamma\right), D^{(1)}\left(\mathrm{H}_1\Gamma\right)\right)$ are in HL_1 because they are images of γ, respectively $(\mathrm{H}_1\Gamma)$, under the action of $\left(\Delta + D^{(1)}\right)$.

Since $\left(\Delta + D^{(1)}\right)(\mathrm{H}_1\Gamma) = \delta'\left(D^{(1)}\left(\mathrm{H}_1\Gamma\right), 0\right)$, the former is exact in HL_1, contradicting the assumption. □

9.7 The connected part

Proposition 9.8. *The complex* $\left(\mathrm{H}_{\geq 1}\mathrm{fHGCc}^{\ddagger}_{-1,1}, \delta + \Delta\right)$ *is acyclic.*

Proof. The complex splits as:

$$\left(\mathrm{H}_{\geq 1}\mathrm{fHGCc}^{\ddagger}_{-1,1}, \delta + \Delta\right) = \prod_{f \in \mathbb{Z}} \left(\mathrm{F}_f \mathrm{H}_{\geq 1}\mathrm{fHGCc}^{\ddagger}_{-1,1}, \delta + \Delta\right).$$

In each of the subcomplexes the degree $d = v + 1 - h = e + 1 - f$ is determined by the number of edges e. We will prove the proposition simultaneously for all subcomplexes by the induction on the number of edges e, i.e. that $H_{e+1-f}\left(\mathrm{F}_f \mathrm{H}_{\geq 1}\mathrm{fHGCc}^{\ddagger}_{-1,1}\delta + \Delta\right) = 0$. For $e = 0$ it is clear. Let us suppose that the claim holds for every number of edges less than e. We prove the claim for e edges.

Suppose the opposite, that there is a class represented by Γ with e edges. By Proposition 9.7 there is $\gamma \in \mathrm{H}_{\geq 1}\mathrm{fHGC}^{\ddagger}_{-1,1}$ such that $\Gamma = (\delta + \Delta)(\gamma)$. It clearly has $e - 1$ edges.

Since in $\mathrm{H}_{\geq 1}\mathrm{fHGC}^{\ddagger}_{-1,1}$ every connected component has at least one edge, the number of connected components is bounded. We write $\gamma = \sum_{i=1}^{k} \mathrm{C}_i\gamma$ where $\mathrm{C}_i\gamma$ is the part with i connected components. Choose γ such that k is minimal possible. If $k = 1$ we are done, so suppose that $k > 1$.

Let Δ_0 be the part of Δ that does not connect two connected components. We now switch to the extended complexes and write:

$$\left(\mathrm{C}_k\mathrm{fHGC}^{\ddagger}_{-1,1}, \delta + \Delta_0\right) = \left(\left(\mathrm{fHGCc}^{\ddagger}_{-1,1}, \delta + \Delta\right)^{\otimes k}\right)^{S_k} [k-1].$$

Taking cohomology commutes with the symmetric product, so

$$H\left(\mathrm{C}_k\mathrm{fHGC}^{\ddagger}_{-1,1}, \delta + \Delta_0\right) = \left(H\left(\mathrm{fHGCc}^{\ddagger}_{-1,1}, \delta + \Delta\right)^{\otimes k}\right)^{S_k} [k-1].$$

In particular, because of the assumption of the induction, cohomology lives in the hairless part for less than e edges.

It holds that $\mathrm{C}_k\gamma \in \mathrm{C}_k\mathrm{fHGC}^{\ddagger}_{-1,1}$ and $(\delta + \Delta_0)\mathrm{C}_k\gamma = 0$. Since $\mathrm{C}_k\gamma$ has $e - 1$ edges, there is $\gamma' \in \mathrm{C}_k\mathrm{fHGC}^{\ddagger}_{-1,1}$ such that $(\delta + \Delta_0)\gamma' - \mathrm{C}_k\gamma$ is hairless.

Now $\gamma - (\delta + \Delta)\gamma'$ is also mapped to Γ by $(\delta + \Delta)$. Δ does not act on hairless part, so the part with k connected components $(\delta + \Delta_0)\gamma' - \mathrm{C}_k\gamma$ can be removed and the resulting element is still mapped to Γ by $(\delta + \Delta)$. It has less than k connecting components, contradicting the minimality of k. □

9.8 The end of the proof

The following proposition finishes the proof of the first part of Theorem 8.3.

Proposition 9.9. *The complex* $(\mathrm{HGC}_{-1,1}, \delta + \Delta)$ *is acyclic.*

Proof. Recall that $\mathrm{HGC}_{-1,1} = \mathrm{H}_{\geq 1}\mathrm{fHGCc}^{\geq 3}_{-1,1}$. On the mapping cone of the inclusion $(\mathrm{HGC}_{-1,1}, \delta + \Delta) \to \left(\mathrm{H}_{\geq 1}\mathrm{fHGCc}^{\ddagger}_{-1,1}, \delta + \Delta\right)$ we set up a spectral sequence on the number of hairs. The discussion from the proof of Proposition 9.6 implies that the spectral sequence converges correctly.

On the first page there is the mapping cone of the inclusion $(\mathrm{HGC}_{-1,1}, \delta) \to \left(\mathrm{H}_{\geq 1}\mathrm{fHGCc}^{\ddagger}_{-1,1}, \delta\right)$. It is acyclic by Corollary 2.27, finishing the proof. □

9.9 A Picture of the cohomology

Like in Section 7.2 we may again use the two spectral sequences to generate many non-trivial hairy cohomology classes out of bald classes in the "waterfall mechanism". The *first* spectral sequence arises from the differential $\delta + \Delta$ of Theorem 8.3 and the *second* one arises from $\delta + \chi$ of Theorem 2.31. The cancellations in the two spectral sequences are illustrated in Figure 9.5 in the computer accessible regime. Note that actual m and n in the figure are different than what we studied here, but of the same parity, leading only to some degree shifting.

	1	2	3	4	5	6	7	8
9	1_8							
8		1_7	1_7					
7			1_8	1_6				
6		1_6	1_6	$1_6, 1_7$				
5	1_4		1_5	$1_5, 1_7$	2_5			
4		1_3		$1_5, 1_6$	1_6			
3			1_4		1_4			
2		1_2			1_5			
1			1_1		1_1	1_4	1_1	$1_1, 1_4$

Figure 9.5: Computer generated table of the dimensions of the hairy graph cohomology $\dim H(\mathrm{HGC}_{3,3})$. The rows indicate the number of hairs (↑), the columns the genus (→). A table entry 1_3 means that there the degree 3 subspace is one-dimensional. The arrows indicate (some of) the cancellations of classes in the two spectral sequences. The computer program used approximate (floating point) arithmetic, so the displayed numbers should not be considered as rigorous results.

Chapter 10

Extra differential for hairy graph complex, *m* odd, *n* even

In this chapter we prove the second part of Theorem 8.3, i.e. that $H\left(\mathrm{HGC}^{\oslash}_{-1,0}, \delta + \Delta\right)$ is one-dimensional, the class being represented by the star σ_3. The proving strategy is very similar to the one in the previous chapter.

$$
\begin{array}{c}
\Big\downarrow 10.4 \\
\left(\mathrm{fHGC}^{\oslash}_{-1,0}, \Delta\right) \text{ is almost acyclic, except for the classes without hairs} \\
\Big\downarrow 10.6 \\
\left(\mathrm{H}_\flat\mathrm{fHGC}^{\oslash}_{-1,0}, \Delta\right) \text{ is almost acyclic} \\
\Big\downarrow 10.7 \\
\left(\mathrm{H}_\flat\mathrm{fHGC}^{\oslash}_{-1,0}, \delta + \Delta\right) \text{ is almost acyclic} \\
\Big\downarrow 10.9 \\
\left(\mathrm{H}_\flat\mathrm{bHGC}^{\oslash}_{-1,0}, \delta + \Delta\right) \text{ is almost acyclic} \\
\Big\downarrow 10.15 \\
\left(\mathrm{H}_{\geq 1}\mathrm{bHGCc}^{\oslash}_{-1,0}, \delta + \Delta\right) \text{ is almost acyclic} \\
\Big\downarrow 10.19 \\
\left(\mathrm{HGC}^{\oslash}_{-1,0}, \delta + \Delta\right) \text{ is almost acyclic}
\end{array}
$$

The problem this time is that $\left(\mathrm{H}_{\geq 1}\mathrm{bHGCc}^{\oslash}_{-1,0}, \delta + \Delta\right)$ is not fully acyclic, there is a class represented by the star σ_3. This makes the complex with disconnected graphs $\left(\mathrm{H}_{\geq 1}\mathrm{bHGC}^{\oslash}_{-1,0}, \delta + \Delta\right)$ far from acyclic. There are not only classes represented by graphs whose connected components are stars σ_3, but also any representative of a class in $H\left(\mathrm{fGCc}_0^{\geq 1\,\oslash}\right)$ with at least one connected component σ_3, to make the whole graph 'hairy'. The second part of Proposition 10.15 shows that this cohomology is nicely governed by the hairless graph cohomology.

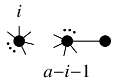
Therefore, the result that the complex with special hairless part $\left(\mathrm{H}_\flat\mathrm{bHGC}^{\oslash}_{-1,0}, \delta + \Delta\right)$ is almost acyclic (Proposition 10.9) may be surprising. It seems that all disconnected classes are cancelled with the hairless part. This is exactly what happens, as shown in Proposition 10.15.

Recall that for $m = -1$ and $n = 0$ the degree is $d = e + 1$.

10.1 The differential Δ

Let for $a \geq 2$

$$\rho_a := \sum_{i=1}^{a-1} \frac{(-1)^i}{i!(a-1-i)!} \sigma_i \cup \lambda_{a-i} = \sum_{i=1}^{a-1} \frac{(-1)^i}{i!(a-1-i)!} \quad \text{(10.1)}$$



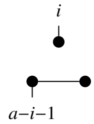
$\in \mathrm{fHGC}_{-1,0}$.

Lemma 10.1. *For even $a \geq 2$ it holds that*

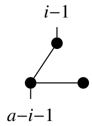
$$\Delta(\rho_a) = 0.$$

Proof.

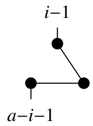
$$\Delta(\rho_a) = \sum_{i=1}^{a-1} \frac{(-1)^i}{i!(a-i-1)!} \Delta\left(\quad \right)$$



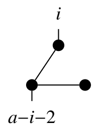
$$= \sum_{i=1}^{a-1} \frac{(-1)^i}{(i-1)!(a-i-1)!}$$



$$+ \sum_{i=1}^{a-1} \frac{(-1)^i}{(i-1)!(a-i-1)!}$$



$$+ \sum_{i=1}^{a-2} \frac{(-1)^i}{i!(a-i-2)!}$$



.

The first and the third term cancel each other, while the second term is easily seen to be 0 for even a, concluding the proof. □

Definition 10.2. Let $c : \mathrm{H}_h\mathrm{fHGC}_{-1,0} \to \mathrm{H}_{h-1}\mathrm{fHGC}_{-1,0}$ be defined on graph Γ as

$$c(\Gamma) = \sum_{x \in V(\gamma)} h(x) c_x(\Gamma) \quad \text{(10.2)}$$

where $h(x)$ is the number of hairs on the vertex x and $c_x(\Gamma)$ deletes one hair at x and adds an antenna like a_x.

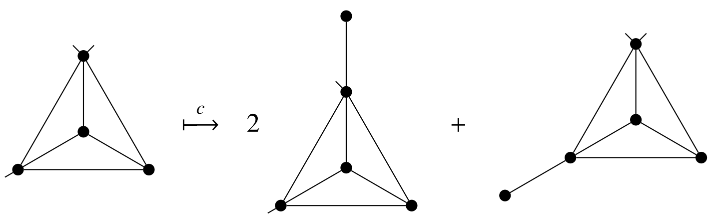


Figure 10.1: Example of the action c, transforming a hair into an antenna.

Lemma 10.3. *For even $a \geq 2$ it holds that*

$$c(\rho_a) = 0.$$

Proof. Because of the symmetry $c(\lambda_a) = 0$ for every $a \geq 1$. Therefore

$$c(\rho_a) = \sum_{i=1}^{a-1} \frac{(-1)^i}{i!(a-1-i)!} c(\sigma_i) \cup \lambda_{a-i} = \sum_{i=1}^{a-1} \frac{(-1)^i}{(i-1)!(a-1-i)!} \lambda_i \cup \lambda_{a-i} = 0.$$

□

The complexes $\left(\mathrm{fHGC}^{\geq 1\,\not\bullet}_{-1,0},\Delta\right)$ and $\left(\mathrm{H}_{\geq 1}\mathrm{fHGC}^{\geq 1\,\not\bullet}_{-1,0},\Delta\right)$ split into the double direct product of complexes, for fixed number of vertices v and for fixed $a = e + h$:

$$\left(\mathrm{fHGC}^{\geq 1\,\not\bullet}_{-1,0},\Delta\right) = \prod_{v\in\mathbb{N}}\prod_{a\in\mathbb{Z}}\left(\mathrm{A}_a\mathrm{V}_v\mathrm{fHGC}^{\geq 1\,\not\bullet}_{-1,0},\Delta\right), \tag{10.3}$$

$$\left(\mathrm{H}_{\geq 1}\mathrm{fHGC}^{\geq 1\,\not\bullet}_{-1,0},\Delta\right) = \prod_{v\in\mathbb{N}}\prod_{a\in\mathbb{Z}}\left(\mathrm{A}_a\mathrm{V}_v\mathrm{H}_{\geq 1}\mathrm{fHGC}^{\geq 1\,\not\bullet}_{-1,0},\Delta\right). \tag{10.4}$$

Proposition 10.4.

- $H\left(\mathrm{A}_a\mathrm{V}_1\mathrm{fHGC}^{\geq 1\,\not\bullet}_{-1,0},\Delta\right)$ *is* 1*-dimensional for* $a \geq 1$*, the class being represented by* σ_a*;*
- $H\left(\mathrm{A}_a\mathrm{V}_2\mathrm{fHGC}^{\geq 1\,\not\bullet}_{-1,0},\Delta\right)$ *is acyclic for even* $a \geq 2$ *and* 1*-dimensional for odd* $a \geq 1$*, the class being represented by* λ_a*;*
- $H\left(\mathrm{A}_a\mathrm{V}_3\mathrm{fHGC}^{\geq 1\,\not\bullet}_{-1,0},\Delta\right)$ *is acyclic for odd* $a \geq 1$ *and* 1*-dimensional for even* $a \geq 2$*, the class being represented by* ρ_a*;*
- $H_{a-h+1}\left(\mathrm{A}_a\mathrm{V}_v\mathrm{fHGC}^{\geq 1\,\not\bullet}_{-1,0},\Delta\right) = 0$ *for* $v \geq 4$ *and* $h \geq 1$.

Note that the last, general claim of the proposition does not say anything about the cohomology at degree $d = a + 1$ and for $d > a + 1$ it is trivially 0.

Proof. For the matter of shortening the notation let

$$W^a_v := \mathrm{A}_a\mathrm{V}_v\mathrm{fHGC}^{\geq 1\,\not\bullet}_{-1,0}, \qquad V^a_v := \mathrm{A}_a\mathrm{V}_v\mathrm{H}_{\geq 1}\mathrm{fHGC}^{\geq 1\,\not\bullet}_{-1,0}. \tag{10.5}$$

We prove the proposition recursively on v. Results for $v = 1, 2$ (first two claims of the proposition) are straightforward.

On W^a_v and V^a_v we choose one vertex and get

$$\dot{W}^a_v := \mathrm{A}_a\dot{\mathrm{V}}_v\mathrm{fHGC}^{\geq 1\,\not\bullet}_{-1,0}, \quad \dot{V}^a_v := \mathrm{A}_a\dot{\mathrm{V}}_v\mathrm{H}_{\geq 1}\mathrm{fHGC}^{\geq 1\,\not\bullet}_{-1,0}, \tag{10.6}$$

the complexes with one vertex chosen, while the others are indistinguishable (see appendix A).

Let us set up a spectral sequence on $\dot{W}^a_v$ on the total valence s of non-chosen vertices, including hairs. So, an edge between non-chosen vertices counts twice, a hair on a non-chosen vertex and an edge between a non-chosen vertex and the chosen vertex counts once, and hairs on the chosen vertex do not count. The differential can not decrease s and splits $\Delta = \Delta_0 + \Delta_1$ where Δ_0 is the part that does not change s. Δ_0 connects a hair from a non-chosen vertex to the chosen vertex and Δ_1 connects an edge to a non-chosen vertex, increasing s always by 1.

For fixed $a = e + h$, s can have only finitely many possible values, so the spectral sequence is finite and converges correctly.

On the first page of the spectral sequence there is the cohomology of $\left(\dot{W}^a_v,\Delta_0\right)$. Let $\chi : \dot{W}^a_v \to \dot{W}^a_v$ be the sum over all edges at the chosen vertex of deleting that edge (as a last edge in numbering, for the matter of sign) and putting a hair on the non-chosen vertex that was connected to that edge, unless it makes the chosen vertex 0-valent, being forbidden by definition. If the chosen vertex is not hairless and 1-valent, it is clear that $\Delta_0\chi + \chi\Delta_0 = C\mathrm{Id}$ where C is the number of edges at the chosen vertex plus the number of hairs on non-chosen vertices. So, a closed graph is exact unless the chosen vertex is isolated with some hairs and there are no other hairs, or it is hairless 1-valent vertex.

Every graph of the form $\Gamma\dot{\cup}\sigma_h$, where $\Gamma \in \mathrm{H}_0 W^{a-h}_{v-1}$, $h \geq 1$ and $\dot{\cup}\sigma_h$ is the operation of gluing the star σ_h that contains the chosen vertex, clearly represents a cohomology class. Let us call it a class of the *first type*. Graphs with hairless 1-valent chosen vertex would not form classes if the vertex would be allowed to be 0-valent. Therefore, cutting that possibility implies that classes are represented by graphs of the form $\Delta_0(\Gamma\dot{\cup}\sigma_0) =: \dot{c}(\Gamma)$ for $\Gamma \in V^x_{v-1}$. Let us call them classes of the *second type*. It is easily seen that $\dot{c}$ is an isomorphism of degree 1, so classes of the second type on the second page of the spectral sequence are indeed equal to the classes of $H(V^a_{v-1}[-1])$. Classes are sketched in Figure 10.2.

As we have already mentioned, on the second page of the spectral sequence in the line of the second-type classes there is a complex isomorphic to $V^a_{v-1}[1]$, the isomorphism being $\dot{c}$ that transforms a hair to an antenna

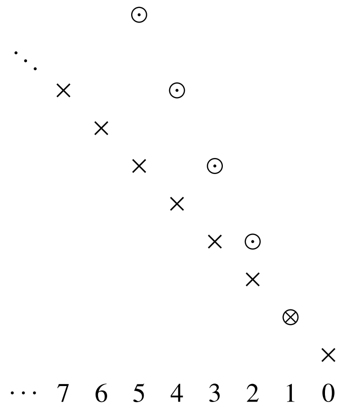


Figure 10.2: Classes on the first page of the spectral sequence of $\dot{W}^a_v$. The numbers at the bottom are the number of hairs h, while the degree is $d = e + 1 = a - h + 1$. Classes of the second type are labelled by $\times$ and classes of the first type are labelled by $\odot$. The position where there are both classes is labelled by $\otimes$.

with the chosen vertex. But there is an "intruder" – for the degree $d = a - 2$ ($h = 1$) there is a class of the first type $\Gamma\dot{\cup}\sigma_1$ in the same position with the class of the second type. All other classes of the first type clearly survive on the second page.

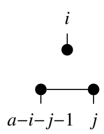
We continue the study first for $v = 3$. It is not possible that $a < 2$. For $a = 2$ it is straightforward that $H\left(W^2_3\right)$ is 1-dimensional, the class being represented by $\rho_2 = -\sigma_1 \cup \lambda_1$.

If $a > 2$ on the first page of the spectral sequence there is only one class of the first type, namely $\lambda_1\dot{\cup}\sigma_{a-1}$ of degree 2. It does not intrude the line with the second type classes and survives until the second page. On the second page in the line of the second-type classes we have the complex isomorphic to $V^a_2[1]$, what is equal to $W^a_2[1]$ in this case. So, there is a class represented by $\dot{c}(\lambda_a)$ of degree 3 for odd a, and no class for even a.

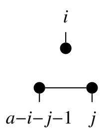
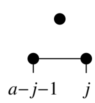
The following lemma shows that for odd a the two classes cancel on further pages, so $\dot{V}^a_3$ is acyclic. Proposition A.2 implies that V^a_3 is acyclic too.

Lemma 10.5. $\left(\mathrm{A}^a\dot{\mathrm{V}}^3\mathrm{fHGC}^{\circledast}_{-1,0}, \Delta\right)$ *is acyclic for odd* $a \geq 3$.

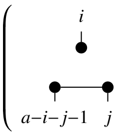
Proof. We still have a spectral sequence on the total valence of non-chosen vertices, including hairs, and get two classes that survive second page: $\lambda_1\dot{\cup}\sigma_{a-1}$ and $\dot{c}(\lambda_a)$. Let

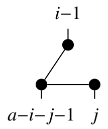
$$\xi := \sum_{i=1}^{a-1}\sum_{j=0}^{a-i-1}(-1)^i\binom{a-1}{i, j, a-i-j-1}$$  $$\in \dot{V}^a_3$$

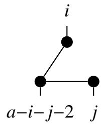
where the upper vertex is chosen. We will work also with

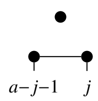
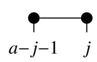
$$\bar{\xi} := \sum_{i=0}^{a-1}\sum_{j=0}^{a-i-1}(-1)^i\binom{a-1}{i, j, a-i-j-1}$$  $$= \xi + \sum_{j=0}^{a-1}\binom{a-1}{j}$$  .

It holds that

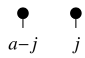
$$\Delta(\bar{\xi}) = \sum_{j=0}^{a-1}\sum_{i=0}^{a-j-1}(-1)^i\binom{a-1}{i,j,a-i-j-1}\Delta\Bigg($$  $$\Bigg)$$

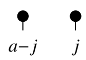
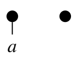
$$= 2\sum_{j=0}^{a-1}\sum_{i=1}^{a-j-1}(-1)^i\frac{(a-1)!}{(i-1)!j!(a-i-j-1)!}$$ 

$$+ 2\sum_{j=0}^{a-1}\sum_{i=0}^{a-j-2}(-1)^i\frac{(a-1)!}{i!j!(a-i-j-2)!}$$  $$= 0,$$

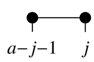
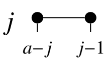
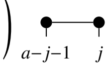
$$\Delta(\xi) = \Delta(\bar{\xi}) - \Delta\left(\sum_{j=0}^{a-1}\binom{a-1}{j}\right.$$  $$\left.\right) = -\dot{c}\left(\sum_{j=0}^{a-1}\binom{a-1}{j}\right.$$  $$\left.\right).$$

Let

$$\nu := \sum_{j=1}^{a-1}\binom{a}{j}$$  $$,$$

$$\bar{\nu} := \sum_{j=0}^{a}\binom{a}{j}$$  $$= \nu + 2$$  $$.$$

It holds that

$$\Delta(\bar{\nu}) = \sum_{j=0}^{a}\frac{a!}{j!(a-j)!}\Bigg((a-j)$$  $$+ j$$  $$\Bigg) = \sum_{j=0}^{a-1}2a\binom{a-1}{j}$$  $$,$$

$$\Delta(\nu) = \Delta(\bar{\nu}) - 2a\lambda_a,$$

$$\Delta(2a\xi + \dot{c}(\nu)) = 2a\Delta(\xi) + \dot{c}(\Delta(\nu)) = -2a\dot{c}(\lambda_a).$$

Element $2a\xi + \dot{c}(\nu)$ contains $\lambda_1\dot{\cup}\sigma_{a-1}$ that survives the second page of the spectral sequence, and the target is exactly multiple of λ_a that also survives, so the two cancel each other. That was to be demonstrated. □

For even a the class in the degree 2 survives. The element $\rho_a \in V_3^a$ is also an element of $\dot{V}_3^a$ where it is equal to the sum over all vertices to be chosen in ρ_a. It contains $\lambda_1\dot{\cup}\sigma_{a-1}$ that forms a class that survives all pages, and since $\Delta(\rho_a) = 0$ (Lemma 10.1), ρ_a represents the class in $H\left(\dot{V}_3^a\right)$. Proposition A.2 implies that it is also a class in $H\left(V_3^a\right)$. That concludes the third claim of the proposition.

For $v \geq 4$ we continue to prove by induction that $H_{a-h+1}\left(\dot{W}_v^a\right) = 0$ for $h > 0$. That will conclude the proof of the proposition. Let us do the step of the induction, while the base ($v = 4$) will be explained later.

Recall that we are not interested in the cohomology $H_{a+1}\left(\dot{W}_v^a\right)$. The second type class in that degree on the first page of the spectral sequence can not kill anything after the second page, so we are not interested what remains there after the second page.

Let C be the complex isomorphic to the line of the second-type classes on the second page of the spectral sequence and it is depicted in Figure 10.3. We split the degree-$(a+1)$ term $\mathrm{H}_1V_{v-1}^a[1]$ into the sum $\mathrm{Ker}_\Delta\left(\mathrm{H}_1V_{v-1}^a\right)\oplus \mathrm{Im}_\Delta\left(\mathrm{H}_1V_{v-1}^a\right)$ and form the 2-row spectral sequence with the intruder $\Gamma\dot{\cup}\sigma_1$, $\Gamma \in \mathrm{H}_0V_{v-1}^{a-1}$, and $\mathrm{Im}_\Delta\left(\mathrm{H}_1V_{v-1}^a\right)$ in the first row. The intruder $\Gamma\dot{\cup}\sigma_1$ is sent to the graph obtained from Γ by adding an antenna in all possible ways. The isomorphic element in $\mathrm{H}_1V_{v-1}^a$ is $\chi^1(\Gamma)$ where χ^1 adds a hair in all possible ways. That element is further split into $\chi_1^1(\Gamma) \in \mathrm{Ker}_\Delta\left(\mathrm{H}_1V_{v-1}^a\right)$ and $\chi_0^1(\Gamma) = \Delta\left(\chi^1(\Gamma)\right) = \nabla(\Gamma) \in \mathrm{Im}_\Delta\left(\mathrm{H}_1V_{v-1}^a\right) \subset \mathrm{H}_0V_{v-1}^a$.

The second row of this spectral sequence of C is the cut version of $W_{v-1}^a[1]$ without the degree-$(a+1)$ part, and its cohomology is equal to $H_{a-h+2}\left(W_{v-1}^a[1]\right)$ for $h > 0$. By the induction hypothesis (for $v > 4$) it is zero, and for the base ($v = 4$) there is for even a a class represented by ρ_a in degree 3. That class survives the spectral

$$\begin{tikzcd}
 & & & & \mathrm{H}_0 V^{a-1}_{v-1} \arrow[r, "\nabla"] \arrow[dr, "\chi^1_1"] & \mathrm{Im}_\Delta(\mathrm{H}_1 V^a_{v-1}) \subset \mathrm{H}_0 V^a_{v-1} \\
\cdots \arrow[r, "\Delta"'] & \mathrm{H}_5 V^a_{v-1} \arrow[r, "\Delta"'] & \mathrm{H}_4 V^a_{v-1} \arrow[r, "\Delta"'] & \mathrm{H}_3 V^a_{v-1} \arrow[r, "\Delta"'] & \mathrm{H}_2 V^a_{v-1} \arrow[r, "\Delta"'] & \mathrm{Ker}_\Delta(\mathrm{H}_1 V^a_{v-1}) \\
d = & a-3 & a-2 & a-1 & a & a+1
\end{tikzcd}$$

Figure 10.3: Complex C split into 2-row spectral sequence.

sequence because even if $a = 2$ it cannot be killed by the first row, since in that case in the first row there is $\mathrm{H}_0 V^1_3 = 0$. So, in any case $H_3(\mathrm{C}) = \mathrm{Ker}_\nabla\left(\mathrm{H}_0 V^{a-1}_{v-1}\right)$, and the rest of the cohomology is zero except for $v = 4$ there is an extra class in $H_3(\mathrm{C})$ being represented by ρ_a.

On the third page of the spectral sequence we have classes of the first type $\Gamma\dot{\cup}\sigma_h$ in degree $d \leq a-1$ and $\mathrm{Ker}_\nabla\left(\mathrm{H}_0 V^{a-1}_{v-1}\right)$ in the degree a. More precisely, in degree a we have classes represented by $\Gamma\dot{\cup}\sigma_1$ for $\Gamma \in \mathrm{Ker}_\nabla\left(\mathrm{H}_0 V^{a-1}_{v-1}\right)$. All classes appear in even s. If $h > 1$ the whole differential $\Delta = \Delta_0 + \Delta_1$ sends the representative of the class as pictured in Figure 10.4.

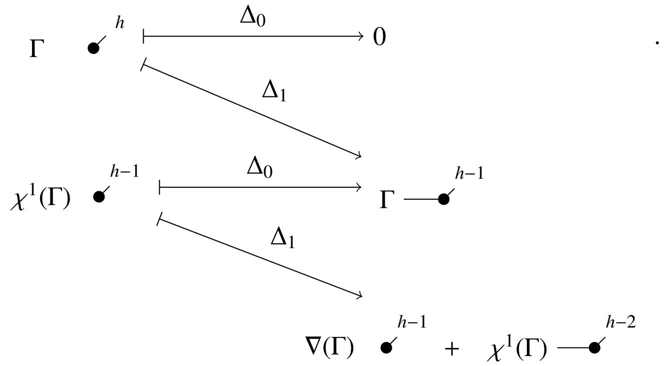


Figure 10.4: Cancelling on the third page of the spectral sequence.

The very last term does not represent a class on the first page (or does not exist when $h = 1$), so on the third page of the spectral sequence there is the differential $\Gamma \cup \sigma_h \mapsto \nabla(\Gamma) \cup \sigma_{h-1}$. Even if $h = 2$ the differential is well defined because it certainly ends in $\mathrm{Ker}_\nabla\left(\mathrm{H}_0 V^{a-1}_{v-1}\right)$. So the complex on the third page looks like in Figure 10.5.

$$\cdots \xrightarrow{\nabla} \mathrm{H}_0 V^{a-5}_{v-1} \xrightarrow{\nabla} \mathrm{H}_0 V^{a-4}_{v-1} \xrightarrow{\nabla} \mathrm{H}_0 V^{a-3}_{v-1} \xrightarrow{\nabla} \mathrm{H}_0 V^{a-2}_{v-1} \xrightarrow{\nabla} \mathrm{Ker}_\nabla\left(\mathrm{H}_0 V^{a-1}_{v-1}\right)$$

Figure 10.5: Third page of the spectral sequence on $\dot{W}^a_v$.

This complex is the cut version of $\left(\mathrm{V}_{v-1}\mathrm{fGC}_0^{\geq 1}, \nabla\right)$, and since $v - 1 \neq 2$ Corollary 5.6 implies that it is acyclic. So, $\dot{V}^a_v$ is acyclic, and because of the proposition A.2 V^a_v is acyclic too. This finishes the step of the induction.

If $v = 4$ and $a > 2$ even (for $a = 2$ the class is out of interest because it is of degree $a+1$), there is one term left that survives on the third page, and therefore till the end: $\dot{c}(\rho_a)$. Lemma 10.1 implies that $\Delta\left(\dot{c}(\rho_a)\right) = \dot{c}(\Delta(\rho_a)) = 0$, so $\dot{c}(\rho_a)$ represents the class. After taking invariants of the action of the symmetric group S_v the class is sent to $c(\rho_a) = 0$ (Lemma 10.3), and Proposition A.2 concludes the base of the induction. $\square$

Note that all classes of $H\left(\mathrm{A}_a\mathrm{fHGC}^{\geq 1}_{-1,0}, \Delta\right)$ are in degrees 1, 2 or $a+1$ (0, 1 or a edges). To simplify the result let us define another complex

$$\mathrm{H}_\flat\mathrm{fHGC}^{\geq 1}_{-1,0} := \mathrm{H}_{\geq 1}\mathrm{fHGC}^{\geq 1}_{-1,0} \oplus \mathrm{HL}_0 \tag{10.7}$$

where

$$\mathrm{HL}_0 := \Delta\left(\mathrm{H}_1\mathrm{fHGC}_{-1,0}^{\geq 1\,\text{⚘}}\right). \tag{10.8}$$

Looking to the subcomplexes $\mathrm{A}_a\mathrm{fHGC}_{-1,0}^{\geq 1\,\text{⚘}}$, we have changed the term with the highest degree $\mathrm{H}_0\mathrm{A}_a\mathrm{fHGC}_{-1,0}^{\geq 1\,\text{⚘}}$ with its subspace HL_0, the image of the differential, to ensure the acyclicity at that degree, as stated in the following corollary.

Corollary 10.6.

$$H\left(\mathrm{H}_\flat\mathrm{fHGC}_{-1,0}^{\geq 1\,\text{⚘}}, \Delta\right) = \begin{cases} [\sigma_a] \textit{ for } a \geq 1, \\ [\lambda_a] \textit{ for odd } a \geq 3, \\ [\rho_a] \textit{ for even } a \geq 2. \end{cases}$$

10.2 The differential $\delta + \Delta$

Since δ and Δ anti-commute, $\mathrm{H}_\flat\mathrm{fHGC}_{-1,0}^{\geq 1\,\text{⚘}}$ is closed under the operation δ, and the complex $\left(\mathrm{H}_\flat\mathrm{fHGC}_{-1,0}^{\geq 1\,\text{⚘}}, \delta + \Delta\right)$ is a subcomplex of $\left(\mathrm{fHGC}_{-1,0}^{\geq 1\,\text{⚘}}, \delta + \Delta\right)$. We study its cohomology in the following proposition.

Proposition 10.7. *All classes of the cohomology $H\left(\mathrm{H}_\flat\mathrm{fHGC}_{-1,0}^{\geq 1\,\text{⚘}}, \delta + \Delta\right)$ consist of the graphs with at most* 1 *edge, i.e. $H_{e+1}\left(\mathrm{H}_\flat\mathrm{fHGC}_{-1,0}^{\geq 1\,\text{⚘}}, \delta + \Delta\right) = 0$ for $e \geq 2$.*

Proof. We set up a spectral sequence on the number of vertices, such that the first differential is Δ. By Corollary 10.6 the first page of the spectral sequence has only zeros in relevant degrees. After splitting the complex

$$\left(\mathrm{H}_\flat\mathrm{fHGC}_{-1,0}^{\geq 1\,\text{⚘}}, \delta + \Delta\right) = \prod_{f\in\mathbb{Z}} \left(\mathrm{F}_f\mathrm{H}_\flat\mathrm{fHGC}_{-1,0}^{\geq 1\,\text{⚘}}, \delta + \Delta\right)$$

Corollary B.3 implies that the spectral sequence converges correctly, hence the result. □

Remark 10.8. Using Corollary 10.6 and easy constructions, all classes of the cohomology $H\left(\mathrm{H}_\flat\mathrm{fHGC}_{-1,0}^{\geq 1\,\text{⚘}}, \delta + \Delta\right)$ can be written down. However, classes consist of complicated sums, and since we do not need them later, we will not calculate them. Let us say that some classes are represented by α and Σ_j defined in (10.10), as the consequence of Lemma 10.10.

10.3 Bounded complex

Let

$$\mathrm{H}_\flat\mathrm{bHGC}_{-1,0}^{\text{⚘}} := \mathrm{H}_{\geq 1}\mathrm{bHGC}_{-1,0}^{\text{⚘}} \oplus \mathrm{HL}_0 \subset \mathrm{H}_\flat\mathrm{fHGC}_{-1,0}^{\geq 1\,\text{⚘}}, \tag{10.9}$$

where HL_0 is as in (10.8) and $\mathrm{bHGC}_{-1,0}$ is defined in Definition 8.12.

Proposition 10.9. *All classes of the cohomology $H\left(\mathrm{H}_\flat\mathrm{bHGC}_{-1,0}^{\text{⚘}}, \delta + \Delta\right)$ consist of the graphs that have at most 1 edge, i.e. $H_{e+1}\left(\mathrm{H}_\flat\mathrm{bHGC}_{-1,0}^{\text{⚘}}, \delta + \Delta\right) = 0$ for $e \geq 2$.*

Proof. On $\mathrm{H}_\flat\mathrm{fHGC}_{-1,0}^{\geq 1\,\text{⚘}}$ we set up a spectral sequence of three rows: $\mathrm{UR}_{-1,0}$, $\mathrm{uHGC}_{-1,0}^{\text{⚘}}$ and $\mathrm{H}_\flat\mathrm{bHGC}_{-1,0}^{\text{⚘}}$. For the degree $d \geq 3$ ($e \geq 2$) the total complex is acyclic by Proposition 10.7 and in first two rows by Lemma 8.14 and Corollary 8.17 there are only two classes in the degrees 1 and 2 that go to zero by the whole differential, so they can not cancel anything in the third row. That concludes the proof. □

10.4 The morphisms π_f

For $j \geq 1$ we define the following.

$$\Sigma_j := \sum_{\substack{k_i\geq 0\\ \sum_i ik_i=j}} \prod_i \frac{(-1)^{k_i}}{k_i!((2i+1)!)^{k_i}} \bigcup_i \sigma_{2i+1}^{\cup k_i} \in \mathrm{bHGC}_{-1,0}. \tag{10.10}$$

For example,

$$\Sigma_3 = \frac{-1}{6\cdot 6^3} + \frac{1}{6\cdot 5!} + \frac{-1}{7!} .$$

Lemma 10.10. *For every $m \geq 1$ it holds that*

$$(\delta + \Delta)(\Sigma_m) = 0.$$

Proof. In all sums i, i' and j are ≥ 1.

$$\begin{aligned}
\delta(\Sigma_m) &= \sum_{\substack{k_i\geq 0\\ \sum_i ik_i=m}} \prod_i \frac{(-1)^{k_i}}{k_i!((2i+1)!)^{k_i}} \left(\sum_i k_i \delta(\sigma_{2i+1}) \cup \sigma_{2i+1}^{\cup k_i-1} \cup \bigcup_{j\neq i} \sigma_{2j+1}^{\cup k_j} \right) = \\
&= \sum_{\substack{k_i\geq 0\\ \sum_i ik_i=m}} \sum_{\substack{i\\ k_i>0}} \frac{(-1)^{k_i}}{(k_i-1)!((2i+1)!)^{k_i}} \prod_{j\neq i} \frac{(-1)^{k_j}}{k_j!((2j+1)!)^{k_j}} \sum_{h=1}^{i-1} \binom{2i+1}{2h} \cup \sigma_{2i+1}^{\cup k_i-1} \cup \bigcup_{j\neq i} \sigma_{2j+1}^{\cup k_j} = \\
&= -\sum_{\substack{k_i\geq 0\\ \sum_i ik_i=m}} \sum_{\substack{i\\ k_i>0}} \sum_{h=1}^{i-1} \left(\frac{1}{(2h)!(2i-2h+1)!} \right) \\
&\qquad \cup \left(\frac{1}{(k_i-1)!} \left(\frac{-1}{(2i+1)!} \sigma_{2i+1} \right)^{\cup k_i-1} \right) \cup \bigcup_{j\neq i} \left(\frac{1}{k_j!} \left(\frac{-1}{(2j+1)!} \sigma_{2j+1} \right)^{\cup k_j} \right) = \\
&= -\sum_{i,i'} \sum_{\substack{k_j\geq 0\\ i+i'+\sum_j jk_j=m}} \left(\frac{1}{(2i)!(2i'+1)!} \right) \cup \bigcup_j \left(\frac{1}{k_j!} \left(\frac{-1}{(2j+1)!} \sigma_{2j+1} \right)^{\cup k_j} \right),
\end{aligned}$$

$$\begin{aligned}
\Delta(\Sigma_m) &= \sum_{\substack{k_i\geq 0\\ \sum_i ik_i=m}} \prod_i \frac{(-1)^{k_i}}{k_i!((2i+1)!)^{k_i}} \left(\sum_i k_i(k_i-1)(2i+1) \cup \sigma_{2i+1}^{\cup k_i-2} \cup \bigcup_{j\neq i} \sigma_{2j+1}^{\cup k_j} + \right. \\
&\qquad \left. + \sum_{\substack{i,i'\\ i\neq i'}} k_i k_{i'} (2i+1) \cup \sigma_{2i+1}^{\cup k_i-1} \cup \sigma_{2i'+1}^{\cup k_{i'}-1} \cup \bigcup_{j\neq i,i'} \sigma_{2j+1}^{\cup k_j} \right) = \\
&= \sum_{\substack{k_i\geq 0\\ \sum_i ik_i=m}} \sum_{\substack{i\\ k_i\geq 2}} \left(\frac{1}{(2i)!(2i+1)!} \right) \cup \left(\frac{1}{(k_i-2)!} \left(\frac{-1}{(2i+1)!} \sigma_{2i+1} \right)^{\cup k_i-2} \right) \cup \bigcup_{j\neq i} \left(\frac{1}{k_j!} \left(\frac{-1}{(2j+1)!} \sigma_{2j+1} \right)^{\cup k_j} \right) + \\
&\quad + \sum_{\substack{k_i\geq 0\\ \sum_i ik_i=m}} \sum_{\substack{i,i'\\ i\neq i';k_i,k_{i'}\geq 1}} \left(\frac{1}{(2i)!(2i'+1)!} \right) \cup \left(\frac{1}{(k_i-1)!} \left(\frac{-1}{(2i+1)!} \sigma_{2i+1} \right)^{\cup k_i-1} \right) \\
&\qquad \cup \left(\frac{1}{(k_{i'}-1)!} \left(\frac{-1}{(2i'+1)!} \sigma_{2i'+1} \right)^{\cup k_{i'}-1} \right) \cup \bigcup_{j\neq i} \left(\frac{1}{k_j!} \left(\frac{-1}{(2j+1)!} \sigma_{2j+1} \right)^{\cup k_j} \right) = \\
&= \sum_i \sum_{\substack{k_j\geq 0\\ 2i+\sum_j jk_j=m}} \left(\frac{1}{(2i)!(2i'+1)!} \right) \cup \bigcup_j \left(\frac{1}{k_j!} \left(\frac{-1}{(2j+1)!} \sigma_{2j+1} \right)^{\cup k_j} \right) \\
&\quad + \sum_{\substack{i,i'\\ i\neq i'}} \sum_{\substack{k_j\geq 0\\ i+i'+\sum_j jk_j=m}} \left(\frac{1}{(2i)!(2i'+1)!} \right) \cup \bigcup_j \left(\frac{1}{k_j!} \left(\frac{-1}{(2j+1)!} \sigma_{2j+1} \right)^{\cup k_j} \right) = -\delta(\Sigma_m).
\end{aligned}$$

□

The coefficients of Σ_j are indeed not surprising, they divide a graph with its order of symmetry (exchanging same stars and hairs in a star) such that coefficients from the operations disappear.

Definition 10.11. Let $\chi^d : \mathrm{H}_h\mathrm{fHGC}_{-1,0} \to \mathrm{H}_{h+d}\mathrm{fHGC}_{-1,0}$, $\chi^d = \left(\chi^1\right)^d$, for $d \geq 2$. We also set $\chi^d = 0$ for $d < 0$.

The map χ^d adds d hairs in all possible ways, but there is a multinomial coefficient $\binom{d}{k_1,k_2,\dots}$ before, where k_i is the number of hairs added to the vertex i.

Lemma 10.12. *For every* $\Gamma \in \mathrm{fGC}_0^{\geq 1\,\bullet}$ *and* $m \geq 0$ *it holds that*

$$\sum_{n=0}^{m-1} \frac{1}{(2n)!}\left(\Delta\left(\chi^{2n}(\Gamma)\cup\Sigma_{m-n}\right) - \Delta\chi^{2n}(\Gamma)\cup\Sigma_{m-n} - \chi^{2n}(\Gamma)\cup\Delta(\Sigma_{m-n})\right) = -\sum_{n=1}^{m}\frac{1}{(2n)!}\left(\delta\chi^{2n}(\Gamma)-\chi^{2n}\delta(\Gamma)\right)\cup\Sigma_{m-n}.$$

$$\sum_{n=1}^{m-1} \frac{1}{(2n-1)!}\left(\Delta\left(\chi^{2n-1}(\Gamma)\cup\Sigma_{m-n}\right) - \Delta\chi^{2n-1}(\Gamma)\cup\Sigma_{m-n} - \chi^{2n-1}(\Gamma)\cup\Delta(\Sigma_{m-n})\right) =$$
$$= -\sum_{n=1}^{m}\frac{1}{(2n-1)!}\left(\delta\chi^{2n-1}(\Gamma)-\chi^{2n-1}\delta(\Gamma)\right)\cup\Sigma_{m-n}.$$

Proof. Let us prove only the first equation, while the second equation is similar. We make notations such that the claim is $\sum L_n = -\sum R_n$. L_n is exactly the part of $\Delta\left(\chi^{2n}(\Gamma)\cup\Sigma_{m-n}\right)$ that connects $\chi^{2n}(\Gamma)$ with Σ_{m-n}. It can be done in two ways, a hair from $\chi^{2n}(\Gamma)$ connects to a star in Σ_{m-n}, or a star from Σ_{m-n} connects to $\chi^{2n}(\Gamma)$. In both cases 'a flower' with 2 or more hairs is added to $\chi^{2n}(\Gamma)$. Let f_k be a map that adds a flower with k hairs to a vertex in all possible ways. In all sums i and j are ≥ 1. It holds that:

$$L_n = \frac{1}{(2n)!}\sum_{\substack{k_j\geq 0\\ \sum_j jk_j=m-n}}\prod_j\frac{(-1)^{k_j}}{k_j!((2j+1)!)^{k_j}}\sum_i k_i\left(2nf_{2i+1}\chi^{2n-1}(\Gamma)+(2i+1)f_{2i}\chi^{2n}(\Gamma)\right)\cup\sigma_{2i+1}^{\cup k_i-1}\cup\bigcup_{j\neq i}\sigma_{2j+1}^{\cup k_j} =$$
$$= -\sum_i\sum_{\substack{k_j\geq 0\\ i+\sum_j jk_j=m-n}}\left(\frac{1}{(2i+1)!(2n-1)!}f_{2i+1}\chi^{2n-1}(\Gamma)+\frac{1}{(2i)!(2n)!}f_{2i}\chi^{2n}(\Gamma)\right)\cup\bigcup_j\left(\frac{1}{k_j!}\left(\frac{-1}{(2j+1)!}\sigma_{2j+1}\right)^{\cup k_j}\right) =$$
$$= -\sum_i\left(\frac{1}{(2i+1)!(2n-1)!}f_{2i+1}\chi^{2n-1}(\Gamma)+\frac{1}{(2i)!(2n)!}f_{2i}\chi^{2n}(\Gamma)\right)\cup\Sigma_{m-n-i}.$$

On the other side there is

$$R_n = \frac{\Sigma_{m-n}}{(2n)!}\cup\sum_x\left(\frac{1}{2}s_x(\chi^{2n}(\Gamma)) - a_x(\chi^{2n}(\Gamma)) - h(x)e_x(\chi^{2n}(\Gamma)) - \frac{1}{2}\chi^{2n}(s_x(\Gamma)) + \chi^{2n}(a_x(\Gamma))\right),$$

where x runs through vertices of Γ. It is clear $s_x(\chi^{2n}(\Gamma)) = \chi^{2n}(s_x(\Gamma))$. In $\chi^{2n}(a_x(\Gamma))$ hairs are added to vertices of Γ or to the new vertex of the antenna. Terms $-a_x(\chi^{2n}(\Gamma))$ and $-h(x)e_x(\chi^{2n}(\Gamma))$ cancel exactly the terms of $\chi^{2n}(a_x(\Gamma))$ where no or one hair is added to the new vertex. So

$$R_n = \frac{\Sigma_{m-n}}{(2n)!}\cup\sum_{i=2}^{2n}\binom{2n}{i}f_i(\chi^{2n-i}(\Gamma)) = \sum_{i=2}^{2n}\frac{1}{i!(2n-i)!}f_i(\chi^{2n-i}(\Gamma))\cup\Sigma_{m-n}.$$

Now a simple play with the sums leads to the result. □

Recall that prefix $\mathrm{B}_{<f,par}$ means all graphs with $e-v<f$ of the same parity as f and that

$$\mathrm{bHGCd}_{-1,0}\subset\mathrm{bHGC}_{-1,0} \tag{10.11}$$

is the subcomplex spanned by disconnected graphs.

Definition 10.13. For every $f\in\mathbb{Z}$ we define degree-0 map $\pi_f : \mathrm{B}_{<f,par}\mathrm{fGC}_0^{\geq 1\,\bullet}[-1]\to\mathrm{F}_f\mathrm{H}_{\geq 1}\mathrm{bHGCd}_{-1,0}^{\bullet}$. Let $\Gamma\in\mathrm{B}_b\mathrm{fGC}_0^{\geq 1}$ for $b<f$ of the same parity as f. Then

$$\pi_f(\Gamma) = \sum_{i=0}^{\frac{m-b}{2}-1}\frac{1}{(2i)!}\chi^{2i}(\Gamma)\cup\Sigma_{\frac{f-b}{2}-i} - \sum_{i=1}^{\frac{f-b}{2}-1}\frac{1}{(2i-1)!}\chi^{2i-1}\bar{D}(\Gamma)\cup\Sigma_{\frac{f-b}{2}-i}. \tag{10.12}$$

Recall that $\tilde{\delta} = e^{\bar{D}}(\delta+\nabla)e^{-\bar{D}} = \delta+\bar{D}\nabla$ (5.17, (5.1)).

Lemma 10.14. *For every* $\Gamma \in \mathrm{B}^b\mathrm{fGC}_0^{\geq 1\,\varnothing}$ *and* $m \in \mathbb{Z}$ *of the same parity as b it holds that*

$$(\delta + \Delta)\pi_m(\Gamma) = \pi_m\tilde{\delta}(\Gamma)$$

in the complex $\mathrm{H}_{\geq 1}\mathrm{bHGC}_{-1,0}^{\varnothing}$.

Proof. The left-hand side is

$$(\delta + \Delta)\pi_m(\Gamma) = (\delta + \Delta)\left(\sum_{n=0}^{\frac{m-b}{2}-1} \frac{1}{(2n)!}\chi^{2n}(\Gamma) \cup \Sigma_{\frac{m-b}{2}-n} - \sum_{n=1}^{\frac{m-b}{2}-1} \frac{1}{(2n-1)!}\chi^{2n-1}\bar{D}(\Gamma) \cup \Sigma_{\frac{m-b}{2}-n}\right) =$$

$$= \sum_{n=0}^{\frac{m-b}{2}-1}\left(\frac{\delta\left(\chi^{2n}(\Gamma)\right)}{(2n)!} \cup \Sigma_{\frac{m-b}{2}-n} + \frac{\chi^{2n}(\Gamma)}{(2n)!} \cup \delta\Sigma_{\frac{m-b}{2}-n}\right)$$

$$- \sum_{n=1}^{\frac{m-b}{2}-1}\left(\frac{\delta\left(\chi^{2n-1}\bar{D}(\Gamma)\right)}{(2n-1)!} \cup \Sigma_{\frac{m-b}{2}-n} + \frac{\chi^{2n-1}\bar{D}(\Gamma)}{(2n-1)!} \cup \delta\Sigma_{\frac{m-b}{2}-n}\right)$$

$$+ \sum_{n=0}^{\frac{m-b}{2}-1} \frac{\Delta\left(\chi^{2n}(\Gamma) \cup \Sigma_{\frac{m-b}{2}-n}\right)}{(2n)!} - \sum_{n=1}^{\frac{m-b}{2}-1} \frac{\Delta\left(\chi^{2n-1}\bar{D}(\Gamma) \cup \Sigma_{\frac{m-b}{2}-n}\right)}{(2n-1)!}.$$

Using Lemmas 10.10 and 10.12, Corollary 5.16 and a clear fact that $\Delta\chi^n(\Gamma) = n\chi^{n-1}\nabla(\Gamma)$ it follows:

$$(\delta + \Delta)\pi_m(\Gamma) = \sum_{n=0}^{\frac{m-b}{2}-1}\left(\frac{\chi^{2n}\delta(\Gamma)}{(2n)!} \cup \Sigma_{\frac{m-b}{2}-n} + \frac{\chi^{2n}(\Gamma)}{(2n)!} \cup \delta\Sigma_{\frac{m-b}{2}-n}\right)$$

$$- \sum_{n=1}^{\frac{m-b}{2}-1}\left(\frac{\chi^{2n-1}\delta\bar{D}(\Gamma)}{(2n-1)!} \cup \Sigma_{\frac{m-b}{2}-n} + \frac{\chi^{2n-1}\bar{D}(\Gamma)}{(2n-1)!} \cup \delta\Sigma_{\frac{m-b}{2}-n}\right)$$

$$+ \sum_{n=0}^{\frac{m-b}{2}-1} \frac{\Delta\chi^{2n}(\Gamma) \cup \Sigma_{\frac{m-b}{2}-n} + \chi^{2n}(\Gamma) \cup \Delta\Sigma_{\frac{m-b}{2}-n}}{(2n)!}$$

$$- \sum_{n=1}^{\frac{m-b}{2}-1} \frac{\Delta\chi^{2n-1}\bar{D}(\Gamma) \cup \Sigma_{\frac{m-b}{2}-n} + \chi^{2n-1}\bar{D}(\Gamma) \cup \Delta\Sigma_{\frac{m-b}{2}-n}}{(2n-1)!} =$$

$$= \sum_{n=0}^{\frac{m-b}{2}-1}\left(\frac{\chi^{2n}\delta(\Gamma)}{(2n)!} \cup \Sigma_{\frac{m-b}{2}-n}\right) - \sum_{n=1}^{\frac{m-b}{2}-1}\left(\frac{\chi^{2n-1}\bar{D}\delta(\Gamma)}{(2n-1)!} \cup \Sigma_{\frac{m-b}{2}-n}\right) + \sum_{n=1}^{\frac{m-b}{2}-1} \frac{\chi^{2n-2}\bar{D}\nabla(\Gamma)}{(2n-2)!} \cup \Sigma_{\frac{m-b}{2}-n}.$$

The right-hand side is

$$\pi_m\tilde{\delta}(\Gamma) = \pi_m(\delta(\Gamma) + \bar{D}\nabla(\Gamma)).$$

The first term under π_m is in $\mathrm{B}_b\mathrm{fGC}_0^{\geq 1}$, and the second one is in $\mathrm{B}_{b+2}\mathrm{fGC}_0^{\geq 1}$, so

$$\pi_m\tilde{\delta}(\Gamma) = \sum_{n=0}^{\frac{m-b}{2}-1} \frac{1}{(2n)!}\chi^{2n}\delta(\Gamma) \cup \Sigma_{\frac{m-b}{2}-n} - \sum_{n=1}^{\frac{m-b}{2}-1} \frac{1}{(2n-1)!}\chi^{2n-1}\bar{D}\delta(\Gamma) \cup \Sigma_{\frac{m-b}{2}-n}$$

$$+ \sum_{n=0}^{\frac{m-b}{2}-2} \frac{1}{(2n)!}\chi^{2n}\bar{D}\nabla(\Gamma) \cup \Sigma_{\frac{m-b}{2}-n-1} - \sum_{n=1}^{\frac{m-b}{2}-2} \frac{1}{(2n-1)!}\chi^{2n-1}\bar{D}\bar{D}\nabla(\Gamma) \cup \Sigma_{\frac{m-b}{2}-n-1} = (\delta + \Delta)\pi_m(\Gamma).$$

□

10.5 The end of the proof

Proposition 10.15.

- *The cohomology* $H\left(\mathrm{H}_{\geq 1}\mathrm{bHGCc}_{-1,0}^{\varnothing}, \delta + \Delta\right)$ *is one-dimensional, the class being represented by the star* σ_3,

- $\pi_f : \left(\mathrm{B}_{<f,par}\mathrm{fGC}_0^{\geq 1\,\not\bullet}[-1], \tilde{\delta}\right) \to \left(\mathrm{F}_f\mathrm{H}_{\geq 1}\mathrm{bHGCd}_{-1,0}^{\not\bullet}, \delta + \Delta\right)$ *is quasi-isomorphism in every degree* $d \geq 2$ $(e \geq 1)$ *for every* $f \in \mathbb{Z}$.

Proof. We prove both statements simultaneously by the induction on the degree $d = e + 1$.

The following lemma shows the first claim for degrees $d = 1$ and 2.

Lemma 10.16.

- $H_1\left(\mathrm{H}_{\geq 1}\mathrm{bHGCc}_{-1,0}^{\not\bullet}, \delta + \Delta\right)$ *is one dimensional, the class being generated by* $\sigma_3 =$.
- $H_2\left(\mathrm{H}_{\geq 1}\mathrm{bHGCc}_{-1,0}^{\not\bullet}, \delta + \Delta\right) = 0$.

Proof. Connected graphs with at most 1 edge are either a star, or a graph with 2 vertices and an edge between them. In both cases, second differential Δ does not do anything. So we need to show the lemma for the complex $\left(\mathrm{H}_{\geq 1}\mathrm{bHGCc}_{-1,0}^{\not\bullet}, \delta\right) := \prod_{h\geq 1}\left(\mathrm{H}_h\mathrm{bHGCc}_{-1,0}^{\not\bullet}, \delta\right)$. For $h \in \{1, 2, 3\}$ the claim can be easily checked by hand. For $h > 3$ the action of the differential δ on the degrees 0 and 1 in the complex with h hairs is depicted in Figure 10.6.

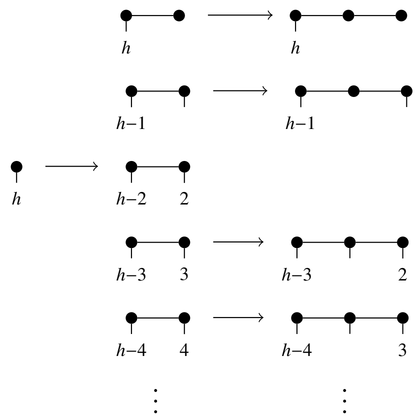


Figure 10.6: The action of the differential δ on the degrees 1 and 2. The differential does not map from the lower row to the higher one. The list goes until the left vertex in degree 2 has more or equal hairs as the right vertex.

One can make a spectral sequence with the rows on the figure, such that the first differential is the one depicted by arrows. This clearly concludes the proof. □

Let $d \geq 2$ and suppose that the first claim holds for all degrees $\leq d$. We want to prove the second claim in degree d.

In general, a complex (C, δ) is acyclic in the degree i if and only if the *cut complex* $F^i(\mathrm{C}, \delta)$ defined as

$$\left(F^i(\mathrm{C}, \delta)\right)_j := \begin{cases} \mathrm{C}_i & j = i, \\ \delta(\mathrm{C}_i) & j = i + 1, \\ \mathrm{C}_{i-1}/\mathrm{Ker}(\delta) & j = i - 1, \\ 0 & \text{otherwise}, \end{cases} \tag{10.13}$$

is acyclic. Similarly, a map of complexes $\phi : (\mathrm{C}, \delta) \to (\mathrm{D}, \delta)$ is a quasi-isomorphism in degree i if the induced map $\phi : F^i(\mathrm{C}, \delta) \to F^i(\mathrm{D}, \delta)$ is a quasi-isomorphism. So, we need to prove that $\pi_f : F^e\left(\mathrm{B}_{<f,par}\mathrm{fGC}_0^{\geq 1\,\not\bullet}[-1], \tilde{\delta}\right) \to F^e\left(\mathrm{F}_f\mathrm{H}_{\geq 1}\mathrm{bHGCd}_{-1,0}^{\not\bullet}, \delta + \Delta\right)$ is a quasi-isomorphism. We do that by proving that the mapping cone is acyclic.

Recall that $\mathrm{C}_{<x}\mathrm{C}$ is the subcomplex of the graph complex C with the number of connected components c smaller than x. Similarly, $\mathrm{P}_{<p}\mathrm{C}$ is the subcomplex of the graph complex C with $2c - e + v < p$. On $\mathrm{B}_{<f,par}\mathrm{fGC}_0^{\geq 1\,\not\bullet}[-1]$ and $\mathrm{F}_f\mathrm{H}_{\geq 1}\mathrm{bHGCd}_{-1,0}^{\not\bullet}$ we consider the filtrations on x: $\mathrm{P}_{\leq 2x-f}\mathrm{B}_{<f,par}\mathrm{fGC}_0^{\geq 1\,\not\bullet}$, respectively $\mathrm{C}_{\leq x}\mathrm{F}_f\mathrm{H}_{\geq 1}\mathrm{bHGCd}_{-1,0}^{\not\bullet}$.

Lemma 10.17. *For every* $f \in \mathbb{Z}$ *the map* π_f *respects the filtrations* $\mathrm{P}_{\leq 2x-f}\mathrm{B}_{<f,par}\mathrm{fGC}_0^{\geq 1\,\not\bullet}$ *and* $\mathrm{C}_{\leq x}\mathrm{F}_f\mathrm{H}_{\geq 1}\mathrm{bHGCd}_{-1,0}^{\not\bullet}$, *i.e. for every* $x \in \mathbb{Z}$ $\pi_f\left(\mathrm{P}_{\leq 2x-m}\mathrm{B}_{<f,par}\mathrm{fGC}_0^{\geq 1\,\not\bullet}\right) \subset \mathrm{C}_{\leq x}\mathrm{F}_f\mathrm{H}_{\geq 1}\mathrm{bHGCd}_{-1,0}^{\not\bullet}$.

Proof. Let $\Gamma \in \mathrm{P}_{\leq 2x-f}\mathrm{B}_{<f,par}\mathrm{fGC}_0^{\geq 1\,\varphi}$ be a graph with v vertices, e edges and c connected components. All those constraints mean that $e-v$ is of the same parity as f and $<f$, and $2c-v+e \leq 2x-f$.

Σ_j has at most j connected components. Therefore, the first sum in $\pi_f(\Gamma)$ has the most number of connected components for $n=0$ and it is $c+\frac{f-e+v}{2} \leq x$. The same is true for the second sum because $\bar{D}$ can not increase the number of connected components. □

Actually, we set up a spectral sequence on the cut version of complexes. The lemma implies that those filtrations induce a filtration of the mapping cone. One checks that the filtration is bounded above, so it converges correctly. It is now enough to show that the complex with the first differential is acyclic. The complex with the first differential is the mapping cone of $\pi_f^0 : F^e\left(\mathrm{B}_{<f,par}\mathrm{fGC}_0^{\geq 1\,\varphi}[-1], \delta\right) \to F^e\left(\mathrm{F}_f\mathrm{H}_{\geq 1}\mathrm{bHGCd}_{-1,0}^{\varphi}, \delta+\Delta_0\right)$ where Δ_0 is the part of the differential that does not change the number of connected components and $\pi_f^0 :$ $\mathrm{B}_{<f,par}\mathrm{fGC}_0^{\geq 1\,\varphi}[-1] \to \mathrm{F}_f\mathrm{H}_{\geq 1}\mathrm{bHGC}_{-1,0}^{\varphi}$ is

$$\pi_f^0(\Gamma) = C\,\Gamma \cup \sigma_3^{\cup\frac{f-b}{2}}, \tag{10.14}$$

where C is an irrelevant coefficient. The acyclicity of this mapping cone is equivalent to $\pi_f^0 : \left(\mathrm{B}_{<f,par}\mathrm{fGC}_0^{\geq 1\,\varphi}[-1], \delta\right) \to \left(\mathrm{F}_f\mathrm{H}_{\geq 1}\mathrm{bHGCc}_{-1,0}^{\varphi}, \delta+\Delta_0\right)$ being a quasi-isomorphism in degree d, and that are we going to show. The "trip" to the cut version of the complex was needed only to "save" the degree through the mapping cones.

The assumption says that $H\left(\mathrm{H}_{\geq 1}\mathrm{bHGCc}_{-1,0}^{\varphi}, \delta+\Delta\right)$ is one-dimensional up to degree d, the class being represented by the star σ_3. It implies that $H\left(\mathrm{bHGCc}_{-1,0}^{\varphi}, \delta+\Delta\right)$ is generated by the star σ_3 and classes of $H\left(\mathrm{fGCc}_0^{\geq 1\,\varphi}[-1], \delta\right)$, up to the degree d.

The complex $\left(\mathrm{bHGC}_{-1,0}^{\varphi}, \delta+\Delta_0\right)$ is the symmetric product of $\left(\mathrm{bHGCc}_{-1,0}^{\varphi}, \delta+\Delta\right)$. Since the cohomology commutes with the symmetric product, its cohomology up to the degree d is generated with the classes that are union of hairless classes and any number of $[\sigma_3]$. A class in $H_d\left(\mathrm{H}_{\geq 1}\mathrm{bHGC}_{-1,0}^{\varphi}, \delta+\Delta_0\right)$ has to have at least one star class $[\sigma_3]$, and also a hairless component. Therefore, it also represents a class in the disconnected part $H_d\left(\mathrm{H}_{\geq 1}\mathrm{bHGCd}_{-1,0}^{\varphi}, \delta+\Delta_0\right)$. The union of hairless components is itself a class in $H_d\left(\mathrm{fGC}_0^{\geq 1\,\varphi}[-1], \delta\right)$, and adding stars is exactly the map π_f^0 for some f, so that proves that all classes of $H_d\left(\mathrm{H}_{\geq 1}\mathrm{bHGCd}_{-1,0}^{\varphi}, \delta+\Delta\right)$ come bijectively from classes in $H_d\left(\mathrm{fGC}_0^{\geq 1\,\varphi}[-1], \delta\right)$ and some $f \in \mathbb{Z}$ by the map π_f^0. That proves that $\pi_f^0 :$ $\left(\mathrm{B}_{<f,par}\mathrm{fGC}_0^{\geq 1\,\varphi}[-1], \delta\right) \to \left(\mathrm{F}_f\mathrm{H}_{\geq 1}\mathrm{bHGCd}_{-1,0}^{\varphi}, \delta+\Delta_0\right)$ is quasi-isomorphism at the degree d. That was to be demonstrated for the second claim of the proposition.

Now let $d \geq 3$ and suppose that the second claim holds for the degree $d-1$. We want to prove the first claim in degree d.

On $\left(\mathrm{H}_\flat\mathrm{bHGC}_{-1,0}^{\varphi}, \delta+\Delta\right)$ we set up a spectral sequence of three rows: $\mathrm{H}_{\geq 1}\mathrm{bHGCd}_{-1,0}^{\varphi}$, $\mathrm{H}_{\geq 1}\mathrm{bHGCc}_{-1,0}^{\varphi}$ and HL_0. On the first page there are $H\left(\mathrm{H}_{\geq 1}\mathrm{bHGCd}_{-1,0}^{\varphi}, \delta+\Delta\right)$, $H\left(\mathrm{H}_{\geq 1}\mathrm{bHGCc}_{-1,0}^{\varphi}, \delta+\Delta\right)$ and $H\left(\mathrm{HL}_0, \delta\right)$. Proposition 10.9 implies that $H_d\left(\mathrm{H}_\flat\mathrm{bHGC}_{-1,0}^{\varphi}, \delta+\Delta\right) = 0$, so all classes of the first page in the degree d cancel on further pages.

Let us take a class in the first row $H_{d-1}\left(\mathrm{H}_{\geq 1}\mathrm{bHGCd}_{-1,0}^{\varphi}, \delta+\Delta\right)$. The assumption of the induction implies that it is generated by $\pi_f(\Gamma)$ for some $f \in \mathbb{Z}$ and $\Gamma \in \mathrm{B}_{<f,par}\mathrm{fGC}_0^{\geq 1\,\varphi}[-1]$. Lemma 10.14 implies that $(\delta+\Delta)$ maps that class to 0 in $\mathrm{H}_{\geq 1}\mathrm{bHGC}_{-1,0}^{\varphi}$, so it may only have non-zero part in the third row HL_0. So, all classes of the first row in degree $d-1$ map directly to the third row, and there are no cancellations between first two rows between degrees $d-1$ and e on the second page of the spectral sequence.

Lemma 10.18. *In the spectral sequence of* $\left(\mathrm{H}_\flat\mathrm{bHGC}_{-1,0}^{\varphi}, \delta+\Delta\right)$ *containing rows* $\mathrm{H}_{\geq 1}\mathrm{bHGCd}_{-1,0}^{\varphi}$, $\mathrm{H}_{\geq 1}\mathrm{bHGCc}_{-1,0}^{\varphi}$ *and* HL_0 *classes of* $H\left(\mathrm{H}_{\geq 1}\mathrm{bHGCc}_{-1,0}^{\varphi}, \delta+\Delta\right)$ *and* $H\left(\mathrm{HL}_0, \delta\right)$ *from the first page do not cancel on the second page.*

Proof. Suppose the opposite, i.e. there is $\Gamma \in \mathrm{H}_{\geq 1}\mathrm{bHGCc}_{-1,0}^{\varphi}$ and $\gamma \in \mathrm{HL}_0$ that represent classes in $H\left(\mathrm{H}_{\geq 1}\mathrm{bHGCc}_{-1,0}^{\varphi}, \delta+\Delta\right)$, respectively $H\left(\mathrm{HL}_0, \delta\right)$, and that cancel each other. Therefore, $(\delta+\Delta)\Gamma$ is in the class of γ. We may choose γ such that $(\delta+\Delta)\Gamma = \gamma$.

Let $\Gamma = \sum_{i\geq 1}\mathrm{H}_i\Gamma$ where $\mathrm{H}_i\Gamma$ is the part with i hairs. Propositions 8.8, 8.10 and 8.11 imply

$$\delta D^{(1)}\left(\mathrm{H}_1\Gamma\right) - D^{(1)}\delta\left(\mathrm{H}_1\Gamma\right) = \Delta\left(\mathrm{H}_1\Gamma\right),$$

$$\delta D^{(2)}\left(\mathrm{H}_2\Gamma\right) - D^{(2)}\delta\left(\mathrm{H}_2\Gamma\right) = -D^{(1)}\Delta\left(\mathrm{H}_2\Gamma\right),$$

$$D^{(2)}\Delta\left(\mathrm{H}_3\Gamma\right) = 0$$

and summing all those equalities implies

$$\delta\left(D^{(1)}\left(\mathrm{H}_1\Gamma\right) + D^{(2)}\left(\mathrm{H}_2\Gamma\right)\right) = D^{(1)}\left(\delta\left(\mathrm{H}_1\Gamma\right) - \Delta\left(\mathrm{H}_2\Gamma\right)\right) + D^{(2)}\left(\delta\left(\mathrm{H}_2\Gamma\right) - \Delta\left(\mathrm{H}_3\Gamma\right)\right) + \Delta\left(\mathrm{H}_1\Gamma\right) = \gamma.$$

Lemma 8.6 and Proposition 8.10 imply that $D^{(1)}\left(\mathrm{H}_1\Gamma\right) + D^{(2)}\left(\mathrm{H}_2\Gamma\right) \in \mathrm{HL}_0$, so γ is exact in (HL_0, δ), contradicting the assumption. □

Therefore, a class in the middle row at degree d, i.e. in $H_d\left(\mathrm{H}_{\geq 1}\mathrm{bHGCc}^{\mathbf{?}}_{-1,0}, \delta + \Delta\right)$, can not cancel with anything. Since everything cancels, there can not be a class in $H_d\left(\mathrm{H}_{\geq 1}\mathrm{bHGCc}^{\mathbf{?}}_{-1,0}, \delta + \Delta\right)$. That was to be demonstrated. □

The following corollary with Proposition 10.15 finishes the proof of Theorem 8.3.

Proposition 10.19. *The inclusion* $\left(\mathrm{HGC}^{\mathbf{?}}_{-1,0}, \delta + \Delta\right) \hookrightarrow \left(\mathrm{H}_{\geq 1}\mathrm{bHGCc}^{\mathbf{?}}_{-1,0}, \delta + \Delta\right)$ *is a quasi-isomorphisms.*

Proof. On the mapping cone of the inclusion we set up a spectral sequence on the number $e - v$. After splitting the complexes according to $f = e + h - v$ one checks that in each degree $d = e + 1$ the spectral sequence is bounded, so it converges correctly by Corollary B.3. On the first page of the spectral sequence there is a mapping cone of the inclusion of complexes only with the standard differential δ, so it is acyclic by Proposition 8.13. Therefore the whole mapping cone is acyclic, and the inclusion is a quasi-isomorphism. □

10.6 A Picture of the cohomology

Like in Section 7.2 we may again use the two spectral sequences to generate many non-trivial hairy cohomology classes out of bald classes in the "waterfall mechanism". The *first* spectral sequence arises from the differential $\delta + \Delta$ of Theorem 8.3 and the *second* one arises from $\delta + [\omega, \cdot]$ of Theorem 2.34. The cancellations in the two spectral sequences are illustrated in Figure 10.7 in the computer accessible regime. Note that actual m and n in the figure are different than what we studied here, but of the same parity, leading only to some degree shifting.

	1	2	3	4	5	6	7	8
9		2_{14}						
8	1_{13}	2_{13}						
7		2_{10}	5_{10}	1_8				
6	1_9	2_9	$3_9, 1_7$	4_7				
5		1_6	3_6	6_6	1_4			
4	1_5	1_5	2_5	$3_5, 1_3$	$4_5, 3_3$			
3		1_2	1_2	3_2	4_2			
2		1_1	1_1	2_1	2_1			
1			1_{-2}	1_{-2}	2_{-2}	$2_{-2}, 1_{-5}$	1_{-5}	

Figure 10.7: Computer generated table of the dimensions of the hairy graph cohomology $\dim H(\mathrm{HGC}_{1,2})$. The rows indicate the number of hairs ($\uparrow$), the columns the genus ($\rightarrow$). A table entry 1_3 means that there the degree 3 subspace is one-dimensional. The arrows indicate (some of) the cancellations of classes in the two spectral sequences. The computer program used approximate (floating point) arithmetic, so the displayed numbers should not be considered as rigorous results.

Appendix A

Group action

In this appendix we clarify one way of calculating cohomology of a graph complex, by doing so first with distinguishing some or all vertices.

Recall that graph spaces are defined as the space of invariants of a finite group that permutes elements of a graph (Definition 2.3).

The space of invariants of the action ρ of finite group G on the space V is

$$V^G = \left\{\gamma \in V | \rho_g(\gamma) = \gamma \text{ for all } g \in G\right\}. \tag{A.1}$$

Let (V, d) be a complex and G a finite group acting on C by the action of degree 0 $\rho_g : \mathrm{C} \to \mathrm{C}$ for $g \in G$. Let the action and the differential commute, i.e.

$$\rho_g d(\gamma) = d\rho_g(\gamma) \tag{A.2}$$

for every $g \in G$ and $\gamma \in \mathrm{C}$. The action of the group can be extended to cohomology of C as $\rho_g([\gamma]) = [\rho_g(\gamma)]$ for $[\gamma] \in H(\mathrm{C})$.

Proposition A.1.

$$H\left(\mathrm{C}^G, d\right) = H(\mathrm{C}, d)^G.$$

Proof. Straightforward verification. □

In particular, we are interested in the graph complex (C, d) where d does not change the number of vertices, i.e.

$$(\mathrm{C}, d) = \prod_v (\mathrm{V}_v\mathrm{C}, d). \tag{A.3}$$

In those cases differential d can already be defined on the space with v distinguishable vertices $\bar{\mathrm{V}}_v\mathrm{C}$, where $\mathrm{V}_v\mathrm{C} = (\bar{\mathrm{V}}_v\mathrm{C})^{S_v}$ and the differential on $\mathrm{V}_v\mathrm{C}$ is induced from the one on $\bar{\mathrm{V}}_v\mathrm{C}$. The proposition A.1 gives us an easy tool to calculate the cohomology of the graph complex:

$$H(\mathrm{C}, d) = \prod_v H\left(\bar{\mathrm{V}}_v\mathrm{C}, d\right)^{S_v}. \tag{A.4}$$

Cohomology is often easier to calculate on the spaces $\bar{\mathrm{V}}_v\mathrm{C}$ with distinguishable vertices.

The more interesting use in this thesis will be an intermediate step: distinguishing one vertex and indistinguishing other vertices. On $\bar{\mathrm{V}}_v\mathrm{C}$ there is the action of S_{v-1} that permutes first $v-1$ vertices, while always leaving the last vertex fixed. It is the sub-action of the action of the whole S_v. We define

$$\dot{\mathrm{V}}_v\mathrm{C} := \left(\bar{\mathrm{V}}_v\mathrm{C}\right)^{S_{v-1}}. \tag{A.5}$$

The inclusion $i : \dot{\mathrm{V}}_v\mathrm{C} \hookrightarrow \bar{\mathrm{V}}_v\mathrm{C}$ induces the map $i : H\left(\dot{\mathrm{V}}_v\mathrm{C}\right) \to H\left(\bar{\mathrm{V}}_v\mathrm{C}\right)$. The following proposition states that it is enough to consider classes of $H\left(\dot{\mathrm{V}}_v\mathrm{C}\right)$ in finding $H\left(\mathrm{V}_v\mathrm{C}\right)$.

Proposition A.2.

$$H\left(\mathrm{V}_v\mathrm{C}\right) = \left\{c \in i\left(H\left(\dot{\mathrm{V}}_v\mathrm{C}\right)\right) | \rho_g(c) = c \textit{ for all } g \in G\right\}.$$

Proof. Straightforward verification. □

Appendix B

Spectral sequences

In this appendix we discuss some special feature of spectral sequences needed for graph complexes. See [20] for general theory of spectral sequences.

Graph complexes generally have well defined bases that consists of graphs. All sub-complexes are spanned by some subspace of graphs. Therefore defining a filtration of a complex C is equivalent to defining a decreasing sequence of subsets of graphs. This can be done by assigning a number $k \in \mathbb{Z}$ to each graph, and

$$C \subset \cdots \subset \mathrm{X}_{\geq -1}C \subset \mathrm{X}_{\geq 0}C \subset \mathrm{X}_{\geq 1}C \subset \mathrm{X}_{\geq 2}C \subset \ldots \tag{B.1}$$

is the filtration, where $\mathrm{X}_{\geq k}C$ is the sub-complex of C spanned by graphs that has the assigned number greater or equal to k. That number can be anything defined in Table 2.1, but also an artificial label used only to define particular filtration. For this really to be a filtration, the differential d must not decrease k, that is it acts like

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots & \mathrm{X}_0C_0 \arrow[r, "d^0_{0,0}"] \arrow[dr, "d^1_{0,0}"] \arrow[ddr] \arrow[dddr] & \mathrm{X}_0C_1 \arrow[r, "d^0_{0,1}"] \arrow[dr, "d^1_{0,1}"] \arrow[ddr] \arrow[dddr] & \mathrm{X}_0C_2 & \cdots \\
\cdots & \mathrm{X}_1C_0 \arrow[r] \arrow[dr] \arrow[ddr] & \mathrm{X}_1C_1 \arrow[r] \arrow[dr] \arrow[ddr] & \mathrm{X}_1C_2 & \cdots \\
\cdots & \mathrm{X}_2C_0 \arrow[r] \arrow[dr] & \mathrm{X}_2C_1 \arrow[r] \arrow[dr] & \mathrm{X}_2C_2 & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

where X_kC is the subspace of the complex spanned by graphs with the assigned number exactly k. Note that (X_kC, d^0) is a complex. We say that the filtration and the assigned spectral sequence are *on the number k* given by the prefix.

Assume that the filtration is bounded above, i.e. for every degree d there exist k such that for every $k' < k$ $\mathrm{X}_{k'}C_d = 0$. One can set up a spectral sequence to this filtration. We say that spectral sequence *converges correctly* is it weakly converges to the cohomology of the whole complex.

Proposition B.1. *If for every degree $d \in \mathbb{Z}$ there are only finitely many $k \in \mathbb{Z}$ such that $\mathrm{X}_kC_d \neq 0$, spectral sequence converges correctly.*

Proof. Under condition from the proposition, the filtration is bounded, and [20, Clasical Convergence Theorem 5.51] implies the result. ◻

Proposition B.2. *If the filtration is bounded above and* $\mathrm{X}_k C_d$ *are finitely dimensional for all degrees* $d \in \mathbb{Z}$ *and* $k \in \mathbb{Z}$*, spectral sequence converges correctly.*

Proof. One can see that the finite dimensionality of $\mathrm{X}_k C_d$ implies regularity of the spectral sequence ([20, Definition 5.2.10]). Then [20, Complete Convergence Theorem 5.5.10] implies the result. □

Constraints from Proposition B.2 are often not fulfilled. But sometimes the complex splits as a direct product of sub-complexes that fulfill those constraints, and the result still holds, as shown in the next corollary.

Corollary B.3. *Let the complex* C *have a filtration on the number* k *and let it split as*

$$C = \prod_{b \in Z} \mathrm{B}_b C.$$

If all filtrations $\mathrm{X}_{\leq k} \mathrm{B}_b C_d$ *fulfill conditions of Proposition B.1 or B.2 the spectral sequence on* k *of* C *converges correctly.*

The similar story works for the filtration $\mathrm{X}_{\leq k} C$. In that case we also say that the filtration and the assigned spectral sequence are on the number k given by the prefix.

Appendix C

Program for dimensions of graph spaces

Here is an example of a computer program that calculates dimensions of spaces $\mathrm{V}_v^+\mathrm{E}_e^{-+}\mathrm{H}_0\mathrm{GS}^{\geq 3}$ and $\mathrm{V}_v^+\mathrm{E}_e^{-+}\mathrm{C}_1\mathrm{H}_0\mathrm{GS}^{\geq 3}$ that appear in $\mathrm{fGC}_n^{\geq 3}$ and respectively GC_n for n even, and calculates Euler characteristics of $\mathrm{B}_b\mathrm{fGC}_n^{\geq 3}$ and $\mathrm{B}_b\mathrm{GC}_n$. Programs for other symmetries and constraints are similar.

```
#include <stdio.h>
#include <stdlib.h>
#include <time.h>

#define MS 101
#define MT 151
#define CAS 8

typedef long nr;

nr final[MS][MT][CAS], pol[MS][MT], outm[MS][MT], oute[MS][MT], pre[MS][MT];
nr lcm[MS][MS], gcd[MS][MS];
nr ant[MS][MS][MT], scon[MS][MS][MT];
nr con[MS][MS][MS][MT];
nr pt[MS];
nr inv[MT][CAS];
nr pp[5],ii[CAS],fin,p,p2;
nr ec,ms,mt;
int br,br2;

nr invert (nr, nr);
void writepol (nr[][MT]);
void writemat (nr[][MT]);
void polexp (nr[][MT],nr,nr,short);
void polmult (nr[][MT],nr[][MT],nr,nr,short);
void conn (nr, nr, nr, nr, nr,short);
void euler (nr[][MT]);

int main()
  {
  time_t start,end;
  double dif;
  nr i,j,a,b,s,t,c,zad,fal,sum,cc,sums,sumt,num;
  short sig,cas;

  //Input
  printf("This program counts the number of graphs without odd symmetry on edges, with e edges
  and v vertices such that every vertex is at least 3-valent, possibly with tadpoles, and
  calculates the Euler characteristic for fixed e-v, both for all graphs and only connected
  graphs.\n\nEven\n\n");
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```
printf("e-v is at most:\n");
scanf("%ld",&ec);
printf("%ld\n",ec);
ms=2*ec;
mt=3*ec;
printf("How many prime numbers? (up to %d)\n",CAS);
scanf("%d",&br);
printf("%d\n",br);
printf("Write %d prime numbers greather than %ld:\n",br,mt);
for (cas=0;cas<br;cas++)
  {
  scanf("%ld",&pp[cas]);
  printf("%ld\n",pp[cas]);
  }
printf("How many controlling prime numbers? (up to %d)\n",CAS-br);
scanf("%d",&br2);
printf("%d\n",br2);
printf("Write %d more prime numbers greather than %ld:\n",br2,mt);
br2+=br;
for (cas=br;cas<br2;cas++)
  {
  scanf("%ld",&pp[cas]);
  printf("%ld\n",pp[cas]);
  }
printf("\n");

time (&start);

for (cas=0;cas<br2;cas++){

//inverting
for (i=1;(i<=mt) || (i==4);i++) inv[i][cas]=invert(i,pp[cas]);

//LCM & GCD
for (i=1;i<=ms;i++)
  {
  gcd[i][i]=i;
  lcm[i][i]=i;
  }
for (i=1;i<ms;i++) for (j=i+1;j<=ms;j++)
  {
  gcd[i][j]=gcd[i][j-i];
  gcd[j][i]=gcd[i][j];
  lcm[i][j]=i*j/gcd[i][j];
  lcm[j][i]=lcm[i][j];
  }

//connections
for (a=1;2*a<ms;a++) for (b=a+1;a+b<=ms;b++)
  {
  for (s=0;s<=ms;s++) for (t=0;t<=mt;t++) con[a][b][s][t]=0;
  for (c=1;c*lcm[a][b]<=mt;c++) for (i=0;(i*c*lcm[a][b]<=ms) && ((i+1)*c*lcm[a][b]<=mt);i++)
    {
    sig=((i+1)%2)*(c%2)*(lcm[a][b]%2);
    sig=-2*sig+1;
    con[a][b][i*c*lcm[a][b]][(i+1)*c*lcm[a][b]] =
    (con[a][b][i*c*lcm[a][b]][(i+1)*c*lcm[a][b]] - sig*gcd[a][b]*inv[c][cas]) % pp[cas];
    }
  }

//self connections
for (a=1;2*a<=ms;a++)
```

```
  {
  for (s=0;s<=ms;s++) for (t=0;t<=mt;t++) scon[a][s][t]=0;
  for (c=1;c*a<=mt;c++) for (i=0;(i*c*a<=ms) && ((i+1)*c*a<=mt);i++)
    {
    sig=((i+1)%2)*(c%2)*(a%2);
    sig=-2*sig+1;
    scon[a][i*c*a][(i+1)*c*a] = (scon[a][i*c*a][(i+1)*c*a] - sig*a*inv[c][cas]) % pp[cas];
    }
  }

//antenas
for (a=1;a<=ms;a++)
  {
  for (s=0;s<=ms;s++) for (t=0;t<=mt;t++) ant[a][s][t]=0;
  for (c=1;c*a<=ms;c++) for (i=1;c*a*i<=ms;i++)
    {
    sig=(i%2)*(c%2)*(a%2);
    sig=-2*sig+1;
    ant[a][c*a*i][c*a*i] = (ant[a][c*a*i][c*a*i] + sig*inv[c][cas]) % pp[cas];
    }
  for (c=1;2*c*a<=mt;c++) for (i=1;(c*a*i<=ms) && (c*a*(i+1)<=mt);i++)
    {
    sig=((i+1)%2)*(c%2)*(a%2);
    sig=-2*sig+1;
    ant[a][c*a*i][c*a*(i+1)] = (ant[a][c*a*i][c*a*(i+1)]
    - sig*((inv[2][cas]*a)%pp[cas])*inv[c][cas]) % pp[cas];
    }
  for (c=1;2*c*a<=mt;c++)
    {
    ant[a][c*a][2*c*a] = (ant[a][c*a][2*c*a] + inv[2][cas]*inv[c][cas]) % pp[cas];
    }
  }

//antenas on odd cycles
for (a=1;a<=ms;a+=2)
  {
  for (c=1;c*a<=mt;c++)
    {
    sig=c%2;
    sig=-2*sig+1;
    ant[a][0][c*a] = (ant[a][0][c*a] - (a+1)/2*sig*inv[c][cas]) % pp[cas];
    }
  for (c=1;3*c*a<=mt;c++) for (i=1;(2*c*a*i<=ms) && (c*a*(2*i+1)<=mt);i++)
    {
    sig=c%2;
    sig=-2*sig+1;
    ant[a][2*c*a*i][c*a*(2*i+1)] = (ant[a][2*c*a*i][c*a*(2*i+1)]
    - sig*inv[2][cas]*inv[c][cas]) % pp[cas];
    }
  for (c=1;3*c*a<=ms;c++) for (i=1;(c*a*(2*i+1)<=ms) && (2*c*a*(i+1)<=mt);i++)
    {
    ant[a][c*a*(2*i+1)][2*c*a*(i+1)] = (ant[a][c*a*(2*i+1)][2*c*a*(i+1)]
    + inv[2][cas]*inv[c][cas]) % pp[cas];
    }
  }

//antenas on even cycles
for (a=2;a<=ms;a+=2)
  {
  for (c=1;c*a<=mt;c++)
    {
    ant[a][0][c*a] = (ant[a][0][c*a] - a/2*inv[c][cas]) % pp[cas];
```

```
      }
    for (c=1;c*a/2<=mt;c++)
      {
      sig=((a/2)%2)*(c%2);
      sig=-2*sig+1;
      ant[a][0][c*a/2] = (ant[a][0][c*a/2] - sig*inv[c][cas]) % pp[cas];
      }
    for (c=1;(2*c-1)*a<=mt;c++)
      {
      ant[a][(2*c-1)*a/2][(2*c-1)*a] = (ant[a][(2*c-1)*a/2][(2*c-1)*a] + inv[(2*c-1)][cas])
      % pp[cas];
      }
    for (c=1;c*a/2<=mt;c++) for (i=1;(c*a*i<=ms) && (c*(a*i+a/2)<=mt);i++)
      {
      sig=((a/2)%2)*(c%2);
      sig=-2*sig+1;
      ant[a][c*a*i][c*(a*i+a/2)] = (ant[a][c*a*i][c*(a*i+a/2)] - sig*inv[c][cas]) % pp[cas];
      }
    for (c=1;c*a<=mt;c++) for (i=1;(c*(a*i+a/2)<=ms) && (c*a*(i+1)<=mt);i++)
      {
      ant[a][c*(a*i+a/2)][c*a*(i+1)] = (ant[a][c*(a*i+a/2)][c*a*(i+1)] + inv[c][cas]) % pp[cas];
      }
    }

  //Partitions
  for (i=0;i<=ms;i++)
    {
    pt[i]=0;
    }
  pt[ms]=1;
  zad=ms;
  fal=0;
  for (s=0;s<=ms;s++) for (t=0;t<=mt;t++) pol[s][t]=0;
  pol[0][0]=1;

  while (zad>0)
    {
    cc=1;
    for (a=1;a<=zad;a++) for (i=1;i<=pt[a];i++) cc = (((cc*inv[i][cas])%pp[cas])*inv[a][cas])
    % pp[cas];

    for (s=0;s<=ms;s++) for (t=0;t<=mt;t++) pre[s][t]=0;
    for (a=1;a<=zad;a++) for (b=1;b<=zad;b++) for (s=0;s<=ms;s++) for (t=0;t<=mt;t++) pre[s][t]
    = (pre[s][t] + pt[a]*pt[b]*con[a][b][s][t]) % pp[cas];
    for (a=1;a<=zad;a++) for (s=0;s<=ms;s++) for (t=0;t<=mt;t++) if (pt[a]>1) pre[s][t] =
    (pre[s][t] + pt[a]*(pt[a]-1)/2*scon[a][s][t]) % pp[cas];
    for (a=1;a<=zad;a++) for (s=0;s<=ms;s++) for (t=0;t<=mt;t++) pre[s][t] = (pre[s][t]
    + pt[a]*ant[a][s][t]) % pp[cas];

    polexp (pre,fal,mt,cas);

    sum=ms-fal;
    for (s=sum;s<=ms;s++) for (t=0;t<=mt;t++)
      {
      pol[s][t] = (pol[s][t] + cc*oute[s-sum][t]) % pp[cas];
      }

    if (fal>=zad)
      {
      pt[fal]++;
      zad=fal;
      fal=0;
```

```
      } else
      {
      fal=fal+zad*pt[zad];
      pt[zad]=0;
      zad--;
      fal=fal-zad;
      pt[zad]++;
      }
    }

  //Vakuum
  for (s=0;s<=ms;s++) for (t=0;t<=mt;t++) pre[s][t]=0;

  for (c=1;c*3<=ms;c++) for (i=3;c*i<=ms;i++)
    {
    sig=(c%2)*(i%2);
    sig=-2*sig+1;
    pre[c*i][c*i] = (pre[c*i][c*i] + sig*inv[2][cas]*inv[i][cas]) % pp[cas];
    }
  for (c=1;(2*c-1)*2<=ms;c++) for (i=2;(2*c-1)*i<=ms;i++)
    {
    sig=i%2;
    sig=-2*sig+1;
    pre[(2*c-1)*i][(2*c-1)*(i-1)] = (pre[(2*c-1)*i][(2*c-1)*(i-1)]
    + sig*inv[2][cas]*inv[2*c-1][cas]) % pp[cas];
    }
  for (c=1;2*c*2<=ms;c++) for (i=0;2*c*(i+2)<=ms;i++)
    {
    pre[2*c*(i+2)][2*c*(i+1)] = (pre[2*c*(i+2)][2*c*(i+1)] - inv[4][cas]*inv[c][cas]) % pp[cas];
    pre[2*c*(i+2)][2*c*(i+2)] = (pre[2*c*(i+2)][2*c*(i+2)] - inv[4][cas]*inv[c][cas]) % pp[cas];
    sig=c%2;
    sig=-2*sig+1;
    pre[2*c*(i+2)][c*(2*i+3)] = (pre[2*c*(i+2)][c*(2*i+3)] + sig*inv[2][cas]*inv[c][cas])
    % pp[cas];
    }
  for (c=1;c*3<=ms;c++) for (i=0;c*(2*i+3)<=ms;i++)
    {
    pre[c*(2*i+3)][2*c*(i+1)] = (pre[c*(2*i+3)][2*c*(i+1)] - inv[2][cas]*inv[c][cas]) % pp[cas];
    sig=c%2;
    sig=-2*sig+1;
    pre[c*(2*i+3)][c*(2*i+3)] = (pre[c*(2*i+3)][c*(2*i+3)] + sig*inv[2][cas]*inv[c][cas])
    % pp[cas];
    }
  for (c=1;2*c<=ms;c++) for (i=0;2*c*(i+1)<=ms;i++)
    {
    pre[2*c*(i+1)][2*c*(i+1)] = (pre[2*c*(i+1)][2*c*(i+1)] - inv[4][cas]*inv[c][cas]) % pp[cas];
    }
  for (c=1;2*c<=ms;c++)
    {
    pre[2*c][2*c] = (pre[2*c][2*c] + inv[4][cas]) % pp[cas];
    }
  for (c=1;2*c<=ms;c++)
    {
    pre[2*c][2*c] = (pre[2*c][2*c] + ((inv[2][cas]*inv[c][cas])%pp[cas])*c) % pp[cas];
    }
  for (c=1;2*c-1<=ms;c++)
    {
    pre[2*c-1][2*c-1] = (pre[2*c-1][2*c-1] - inv[2*c-1][cas]*c) % pp[cas];
    }
  for (c=1;2*c<=ms;c++)
    {
    sig=c%2;
```

```
  sig=-2*sig+1;
  pre[2*c][c] = (pre[2*c][c] + sig*inv[2][cas]*inv[c][cas]) % pp[cas];
  }
for (c=1;c<=ms;c++)
  {
  pre[c][0] = (pre[c][0] - inv[c][cas]) % pp[cas];
  }

polexp (pre,ms,mt,cas);

//Final polynomial
polmult (pol,oute,ms,mt,cas);

for (s=0;s<=ms;s++) for (t=0;t<=mt;t++)
  {
  final[s][t][cas]=outm[s][t] % pp[cas];
  if (final[s][t][cas]<0) final[s][t][cas]+=pp[cas];
  }

printf ("%ld completed.\n",pp[cas]);
}

//Putting together
p=1;
for (cas=0;cas<br-1;cas++)
  {
  p*=pp[cas];
  ii[cas]=invert(p,pp[cas+1]);
  }
for (s=0;s<=ms;s++) for (t=0;t<=mt;t++)
  {
  p=1;
  pol[s][t]=final[s][t][0];
  for (cas=0;cas<br-1;cas++)
    {
    p*=pp[cas];
    fin=pol[s][t];
    pol[s][t] = final[s][t][cas+1]-fin;
    for (;pol[s][t]<0;) pol[s][t]+=pp[cas+1];
    pol[s][t] = pol[s][t]*ii[cas] % pp[cas+1];
    pol[s][t] = pol[s][t]*p+fin;
    }
  }
p*=pp[br-1];

//Time & information
time (&end);
dif = difftime (end,start);
printf ("\nCalculation took %.2lf seconds to run.\n", dif);

p2=p;
for (cas=br;cas<br2;cas++) p2*=pp[cas];
printf ("\nEverything is mod %ld = ",p2);
printf ("%ld",pp[0]);
for (cas=1;cas<br2;cas++) printf (" * %ld",pp[cas]);
printf (". Star (*) means that the result is bigger than %ld = ",p);
printf ("%ld",pp[0]);
for (cas=1;cas<br;cas++) printf (" * %ld",pp[cas]);
printf (" or there is an overflow.\n\n");

//Prinring result with disconnected graphs
printf ("\nWith disconnected graphs:\n");
```

```
  writemat (pol);
  euler (pol);

  //Connected part
  for (cas=0;cas<br2;cas++){
  for (s=3;s+3<=ms;s++) for (t=(3*s+1)/2;t+5<=mt;t++) for (i=1;(s*i<=ms) && (t*i<=mt);i++)
    {
    sums=s*i;
    sumt=t*i;
    num=1;
    for (j=1;j<=i;j++)
      {
      if (t%2==0) num = (num*(final[s][t][cas]+j-1)*inv[j][cas]) % pp[cas];
      else num = (num*(final[s][t][cas]-j+1)*inv[j][cas]) % pp[cas];
      }
    if (i>1) final[sums][sumt][cas] = (final[sums][sumt][cas] - num) % pp[cas];
    conn (s,t,sums,sumt,num,cas);
    }

  for (s=0;s<=ms;s++) for (t=0;t<=mt;t++)
    {
    final[s][t][cas]=final[s][t][cas] % pp[cas];
    if (final[s][t][cas]<0) final[s][t][cas]+=pp[cas];
    }
  }

  //Putting together
  p=1;
  for (s=0;s<=ms;s++) for (t=0;t<=mt;t++)
    {
    p=1;
    pol[s][t]=final[s][t][0];
    for (cas=0;cas<br-1;cas++)
      {
      p*=pp[cas];
      fin=pol[s][t];
      pol[s][t] = final[s][t][cas+1]-fin;
      for (;pol[s][t]<0;) pol[s][t]+=pp[cas+1];
      pol[s][t] = pol[s][t]*ii[cas] % pp[cas+1];
      pol[s][t] = pol[s][t]*p+fin;
      }
    }
  p*=pp[br-1];

  //Prinring result for connected graphs
  printf ("\n\nOnly connected graphs:\n");
  writemat (pol);
  euler (pol);

  fflush(stdin);
  getchar();
  return 0;
  }

nr invert (nr x, nr m)
  {
  nr a,b,q,r,lll,ll,l;

  x=x%m;
  if (x<0) x+=m;

  a=m;
```

```
  b=x;
  lll=0;
  ll=1;
  for (;b>1;)
    {
    q=a/b;
    r=a%b;
    a=b;
    b=r;
    l=lll-q*ll;
    lll=ll;
    ll=l;
    }

  ll=ll%m;
  if (ll<0) ll+=m;
  return ll;
  }

void polmult (nr aa[][MT], nr bb[][MT], nr mms, nr mmt,short cas)
  {
  nr i,j,s,t;

  for (s=0;s<=mms;s++) for (t=0;t<=mmt;t++)
    {
    outm[s][t]=0;
    for (i=0;i<=s;i++) for (j=0;j<=t;j++) outm[s][t] += aa[i][j]*bb[s-i][t-j];
    outm[s][t] %= pp[cas];
    }
  }

void polexp (nr aa[][MT], nr mms, nr mmt, short cas)
  {
  nr rad[MS][MT];
  nr min=mmt;
  nr i,s,t;

  for (s=0;s<=mms;s++) for (t=0;t<=mmt;t++)
    {
    if ((aa[s][t]!=0) && (s<min) && (t<min))
      {
      if (s<t) min=t; else min=s;
      }
    oute[s][t]=aa[s][t];
    rad[s][t]=aa[s][t];
    }
  oute[0][0]=1;

  for (i=2;i*min<=mmt;i++)
    {
    polmult (rad,aa,mms,mmt,cas);
    for (s=0;s<=mms;s++) for (t=0;t<=mmt;t++)
      {
      rad[s][t]=(inv[i][cas]*outm[s][t]) % pp[cas];
      oute[s][t] = (oute[s][t] + rad[s][t]) % pp[cas];
      }
    }
  }

void conn (nr ss, nr tt,nr sums,nr sumt,nr numm,short cas)
  {
  nr i,j,s,t,num;
```

```
  s=ss;
  for (t=(3*s+1)/2;t<tt;t++) for (i=1;(sums+s*i<=ms) && (sumt+t*i<=mt);i++)
    {
    num=numm;
    for (j=1;j<=i;j++)
      {
      if (t%2==0) num = (num*(final[s][t][cas]+j-1)*inv[j][cas]) % pp[cas];
      else num = (num*(final[s][t][cas]-j+1)*inv[j][cas]) % pp[cas];
      }
    final[sums+s*i][sumt+t*i][cas] = (final[sums+s*i][sumt+t*i][cas] - num) % pp[cas];
    if (sums+s*i+3<=ms) conn (s,t,sums+s*i,sumt+t*i,num,cas);
    }
  for (s=3;s<ss;s++) for (t=(3*s+1)/2;sumt+t<=mt;t++) for (i=1;(sums+s*i<=ms)
  && (sumt+t*i<=mt);i++)
    {
    num=numm;
    for (j=1;j<=i;j++)
      {
      if (t%2==0) num = (num*(final[s][t][cas]+j-1)*inv[j][cas]) % pp[cas];
      else num = (num*(final[s][t][cas]-j+1)*inv[j][cas]) % pp[cas];
      }
    final[sums+s*i][sumt+t*i][cas] = (final[sums+s*i][sumt+t*i][cas] - num) % pp[cas];
    if (sums+s*i+3<=ms) conn (s,t,sums+s*i,sumt+t*i,num,cas);
    }
  }

void writepol (nr aa[][MT])
  {
  nr contt, conts;
  nr s,t;

  conts=0;

  for (s=0;s<=ms;s++)
    {
    contt=0;

    for (t=0;t<=mt;t++)
      {
      if (aa[s][t]!=0) contt++;
      }

    if (contt>0)
      {
      if (conts==1) printf(" + ");
      conts=1;

      printf("(");
      for (t=0;aa[s][t]==0;t++);
      if (aa[s][t]==-1) printf("-");
      if (abs(aa[s][t])!=1) printf("%ld",aa[s][t]);
      if (t>0) printf("t");
      if (t>1) printf("^%ld",t);
      if ((t==0) && (abs(aa[s][0])==1)) printf ("1");

      for (t++;t<=mt;t++) if (aa[s][t]!=0)
        {
        if (aa[s][t]>0) printf (" + "); else printf(" - ");
        if (abs(aa[s][t])!=1) printf("%d",abs(aa[s][t]));
        if (t>0) printf("t");
        if (t>1) printf("^%ld",t);
```

```
        }
      printf(")");
      if (s>0) printf("s^%ld",s);
      }
    }
  printf("\n");
  }

void euler (nr aa[][MT])
  {
  nr i,j,out,sig;

  printf ("\ne-v : Euler characteristic\n");
  for (i=0;i<=ec;i++)
    {
    if (i<10) printf (" ");
    printf (" %ld : ",i);
    out=0;
    sig=1;
    for (j=0;j<=ms;j++)
      {
      out+=sig*aa[j][i+j];
      sig=-sig;
      }
    printf ("%ld\n",out);
    }
  }

void writemat (nr aa[][MT])
  {
  nr s,t,loo,ct;
  short cas;

  printf ("\n");
  for (t=mt;t>=0;t--)
    {
    printf ("%ld",aa[0][t]);
    for (s=1;s<=ms;s++)
      {
      printf (" ");

      ct=1;
      for (cas=br;cas<br2;cas++) if (aa[s][t]%pp[cas]!=final[s][t][cas]) ct=0;
      if (ct==1)
        {
        for (loo=10;loo>0;loo*=10) if ((aa[s][t]<loo) && (p>loo)) printf (" ");
        printf ("%ld",aa[s][t]);
        }
      else
        {
        for (loo=10;loo>0;loo*=10) if (p>loo) printf (" ");
        printf ("*");
        }
      }
    printf ("\n");
    }
  }
```

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